Real Analysis Preliminary Exam
April 19, 2012

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Please use separate bluebooks for parts A and B, and clearly label each bluebook, indicating its contents.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

Part A

(12) 1. Assume that $S$ is a Lebesgue measurable subset of $\mathbb{R}$ and that $S$ has positive measure. Show the existence of two distinct points $x \in S$ and $y \in S$ such that $x - y$ is rational.

(18) 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 0$ and $f(1) = 1$. Prove or give a counterexample:

   (6) a. If $f'$ exists almost everywhere, then $\int_0^1 f'(x) \, dx = 1$.

   (6) b. If $f$ is absolutely continuous, then $\int_0^1 f''(x) \, dx = 1$.

   (6) b. If $f$ is nondecreasing, then $\int_0^1 f''(x) \, dx = 1$.

(12) 3. Let $E$ be a closed Lebesgue measurable subset of $[0,1]$. Prove or disprove:

   (6) a. If $E$ has measure 0, then $E$ is nowhere dense.

   (6) b. If $E$ is nowhere dense, then $E$ has measure 0.

(12) 4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, and assume that $f' \in L^2(\mathbb{R})$ and that $f(0) = 0$. Show that the following limit exists, and compute its value.

$$\lim_{x \to 0^+} x^{-2/3} f(x)$$
Part B

(12) 5. Suppose that $F: [a, b] \rightarrow \mathbb{R}$ and $G: [a, b] \rightarrow \mathbb{R}$ are absolutely continuous. Show that $FG$ is absolutely continuous.

(20) 6.  

(10) a. Let $g_n$ be a sequence of measurable functions on $\mathbb{R}$ satisfying $|g_n(x)| \leq 1$ for all $n$ and for all $x$. Let $V$ be the set of all $h \in L^1(\mathbb{R})$ such that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h(x)g_n(x)dx = 0.$$ 

Prove that $V$ is a closed subspace of $L^1(\mathbb{R})$.

(10) b. Prove the Riemann-Lebesgue lemma, which states that, if $h \in L^1(\mathbb{R})$, then

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h(x)\sin nx dx = 0.$$ 

You may use any correct proof. One approach can be based on part (a). Hint: If $h$ is the indicator function of an interval, the integral can be computed explicitly.

(14) 7.  

(6) a. State Fubini’s Theorem for $L^1$ functions on measure spaces. Be sure to state the hypotheses and conclusions fully and precisely.

(8) b. Let $f(m,n) \in \mathbb{R}$ for all positive integers $m,n$. Suppose that $\sum_{n=1}^{\infty} |f(m,n)| < \frac{1}{n}$ for each positive integer $n$. Using your statement in (a), prove that $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f(m,n)$. 