Instructions

- Solve all the 7 problems of the exam.
- Write only your “codename” on your Blue Books, NOT your actual name.
- Each problem is worth 20 points, so 140 points is the maximum possible to score on this examination.
- No books, papers, or electronic devices may be used in this examination. In particular, you may not use your own scratch paper.
- All answers must be complete and correct. Partial credit will be awarded sparingly. The scorers must not be expected to supply any missing parts of any answer. In particular, if you use a theorem, state it fully and concisely. Solutions of problems not laid out clearly will be penalized heavily. Do not make additional hypotheses that trivialize a problem.
Problem 1
(a) Is there a finite codimension subspace of a Banach space which is not closed?
(b) Prove that if $\phi : E \to F$ is a continuous linear map between Banach spaces $E$ and $F$ and $\phi(E)$ is of finite codimension, then $\phi(E)$ is closed.
(c) Show that if $F$ is a closed subspace of some Banach space and that it has finite codimension, then it has a topological complement.

Problem 2
Let $u \in L^1(0,1)$ be a nonnegative function. Define $E_n = \int_0^1 x^n u(x) \, dx$. Prove the following inequality:
$$\forall n \geq 0 \quad \forall k \in [0,n] \quad E_{n-k} E_k \leq E_0 E_n.$$
Problem 7

(a) Let \( \{f_n\} \) be a sequence of real valued, continuous functions on \([a, b]\), converging uniformly to a function \( f \). Let \( g \) be of bounded variation on \([a, b]\). Prove that:

\[
\lim_{n \to \infty} \int_a^b f_n \, dg = \int_a^b f \, dg.
\]

(b) Let \( f \) be a real valued, continuous function on \([a, b]\). Let \( \{g_n\} \) be a sequence of functions of bounded variation such that their variations are uniformly bounded:

\[
\exists C \quad \forall n \quad V(g_n, [a, b]) \leq C.
\]

Assume that \( g_n(x) \) converges to \( g(x) \), for some bounded function \( g \) and all \( x \in [a, b] \). Show that:

\[
\lim_{n \to \infty} \int_a^b f \, dg_n = \int_a^b f \, dg.
\]