Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

(12) 1. Let $E \subset \mathbb{R}$ have finite Lebesgue measure. Show that $m(E \cap [x, \infty)) \to 0$ as $x \to \infty$.

(12) 2. Let $f : [0,1] \to \mathbb{R}$ be a continuous function whose derivative exists almost everywhere and satisfies $f' \in L^1([0,1])$. Prove or give a counterexample: $\int_0^1 f'(x) \, dx = f(1) - f(0)$.

(16) 3. Let $E$ be a closed Lebesgue measurable subset of $[0,1]$. Prove or disprove:

(a) If $E$ has measure 0, then $E$ is nowhere dense.

(b) If $E$ is nowhere dense, then $E$ has measure 0.

(12) 4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous, and assume that $f' \in L^2([0,1])$ and that $f(0) = 0$. Show that the following limit exists, and compute its value.

$$\lim_{x \to 0^+} x^{-1/2} f(x)$$
5. Let $K$ be a compact subset of $\mathbb{R}$, and suppose that $f : K \to \mathbb{R}$ and $f_n : K \to \mathbb{R}$, $n = 1, 2, \ldots$, are continuous. Suppose that, for every $x \in K$, $f_{n+1}(x) \leq f_n(x)$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} f_n(x) = f(x)$.

a. Show that $f_n(x) \to f(x)$ uniformly on $K$ as $n \to \infty$.

b. Give an example to show that the compactness of $K$ is necessary.

6. Using an appropriate Fourier series, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

7. Assume that $E$ is a bounded Lebesgue measurable subset of $\mathbb{R}$, and let

$$f(t) = \int_E \cos(tx) \, dx, \quad t \in \mathbb{R}.$$ 

Prove or disprove:

a. $f$ has compact support.

b. $f$ is continuously differentiable.