Manifolds & Topology Prelim
April 16, 2013

There are two sections A and B with three problems (with parts) each. Please explain your work carefully, indicating what results (theorems, lemmas etc.) you are using. Please put the sections A and B in different blue books.

Please put your code (and not your real name) on each blue book.

A1 (a) Define the fundamental group $\pi_1(X, x_0)$ of a path connected space $X$ with basepoint $x_0 \in X$

(b) If $X$ and $Y$ are two such spaces with base $x_0 \in X$ and $y_0 \in Y$. What is the fundamental group of the Cartesian product $X \times Y$ with base point $(x_0, y_0)$?

(c) Let $A = X \times X$, with base point $(x_0, x_0)$. Which of the following groups can be the fundamental group of $A$?

$$Z \oplus Z, \ Z/2 \oplus Z/3 \oplus Z/6, \ Z/2 \oplus Z/2, \ Z/2 \oplus Z/5 \oplus Z/10, \ Z/2 \oplus Z/3$$

Here $Z$ is the integers and $Z/n$ is the integers modulo $n$.

A2. Let $X$ be path-connected, locally path-connected, with base point $x_0 \in X$.

(a) Define a covering space $p : \tilde{X} \to X$.

(b) What is the relationship between the fundamental groups of $\tilde{X}$ and $X$, and the number of elements in the set $p^{-1}(x_0)$?

(c) If $\pi_1(X, x_0)$ is $Z/3 \oplus Z/4$, how many distinct covering spaces are there for $X$?

A3. Let $S^n$ be the $n$-sphere,

$$S^n = \{\vec{v} \mid \vec{v} \in \mathbb{R}^{n+1}, \|\vec{v}\| = 1\}$$

(a) If $n > 0$ and $f : S^n \to S^n$ is continuous, define the degree of $f$, $\deg(f)$.

(b) If $f$ is not onto, what is $\deg(f)$?

(c) If $f(x_0, \ldots, x_n) = (x_n, x_{n-1}, \ldots, x_0)$, what is $\deg f$?

(d) What are the possible $\deg f$ for which $(\deg f)^2 = \deg f$? Give an example for $f$ in each case, for which this is possible.
B1. (a) Let $M^k$ be a smooth (differentiable) manifold. Define the concepts of an immersion and an embedding $f : M^n \to \mathbb{R}^k$.

(b) Show that there is no immersion $f : S^1 \to \mathbb{R}$, $\mathbb{R}$ the real numbers.

(c) Show by example that there may be an embedding $f : M^n \to \mathbb{R}^k$, where $k$ is the lowest possible number, yet there is an embedding $g : M^n \times M^n \to \mathbb{R}^{2k-1}$. (Hint: This is very easy).

B2. (a) State the Gauss Bonnet Theorem for an embedding $M^2 \to \mathbb{R}^3$. Define your terms.

(b) Show that every embedding of the torus $S^1 \times S^1$ in $\mathbb{R}^3$ has points of negative Gaussian curvature and points of positive Gaussian curvature.

B3. Let $\omega = (yz + x)dx + (xz - y)dy + xydz$ be a differential 1-form on the space $\mathbb{R}^3$.

(a) Calculate $d\omega$.

(b) Is there a function $f$ so that $\omega = df$?