Instructions

No books, papers, or electronic devices may be used in this Examination!

Write only your “codename” on your Blue Books, NOT your actual name!

This Examination has two Parts, A and B. Solutions of problems in separate Parts must be written in separate Blue Books! The two Parts may be scored by different Scorers.

The point value for each problem is shown in square brackets. There are four problems in Part A, worth 100 points, and four problems in Part B, worth 100 points, for a total of 200 points. The passing score will be based on your total score.

You may use the result of one problem in another one, even if you have not solved the problem in which the result you are using is proved.

All answers must be complete and correct. Partial credit will be awarded sparingly. The Scorers must not be expected to supply any missing parts of any answer.

Problems in each Part may be done in any order. You must clearly identify where the parts of your answers are. The Scorers will not search at length for answers that are incomplete. Rather, they will assume that you stopped your attempt, unless you clearly indicate where the answer continues.

If you use a Theorem, state it fully and concisely, or identify it clearly. In either case, verifying hypotheses explicitly is essential. Do not use a Theorem if the result you are proving is used to prove that Theorem!

In all cases, give the essential explanations and justifications. A significant part of your work in each problem is really the determination of the crucial points!

Do not make assumptions that trivialize a problem.
Part A

A01 [20 pts]: Let $n > 0$ be an integer. Let $X = S^n$, the $n$-sphere. Suppose $f : S^n \to S^n$ is continuous. Assume, for all $x \in S^n$, that $f(x) \neq -x$.

(a) [10 pts] Show, by example, that $f$ might not have a fixed point.

(b) [10 pts] Show that $f$ is homotopic to a map which has a fixed point.

A02 [30 pts]: Do all of the following:

(a) [10 pts] Let $X$ and $Y$ be topological spaces. Assume that $X \cap Y$ has exactly one point $z$, that there is a contractible neighborhood of $z$ in $X$ and that there is a contractible neighborhood of $z$ in $Y$. Let $n > 0$ be an integer. Write $H_n(X \cup Y)$ in terms of $H_n(X)$ and $H_n(Y)$.

(b) [10 pts] Let $X$ be a topological space. Assume that $H_4(X) = \mathbb{Z} \oplus \mathbb{Z}$ and that $H_3(X) = \mathbb{Z}/2\mathbb{Z}$. Compute $H_4(X;\mathbb{Z}/2\mathbb{Z})$.

(c) [10 pts] Let $X$ be a topological space. Let $n > 0$ be an integer. Write $H_n(X \times S^1)$ in terms of $H_0(X), H_1(X), H_2(X), \ldots$. (Hint: You may use the Kunneth or the Mayer-Vietoris Theorem.)

A03 [20 pts]: Let $X$ and $Y$ be path-connected topological spaces. Let $p : X \to Y$ be a covering map. Let $x \in X$ and let $y := p(x)$.

(a) [10 pts] Show that $p_* : \pi_1(X, x) \to \pi_1(Y, y)$ is injective.

(b) [10 pts] Show that $p_* : H_1(X, x) \to H_1(Y, y)$ need not be injective. (Hint: Remember that $H_1$ is the Abelianization of $\pi_1$.)

A04 [30 pts]: Let $n \in \mathbb{Z}$. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ and give $X$ and $Y$ the relative topology inherited from $\mathbb{R}^n$. Assume that $X$ is compact. Let $f : X \to Y$ be a bijective continuous map. Let $\partial X$ and $\partial Y$ denote the boundaries of $X$ and $Y$ in $\mathbb{R}^n$, respectively. Using Invariance of Domain, show that $f(\partial X) = \partial Y$. 
Part B

B01 [20 pts]: Let $\omega$ be the smooth 2-form on $\mathbb{R}^3$ given by: $\omega := -dx \wedge dy - dy \wedge dz + dx \wedge dz$

(a) [5 pts] Show that $d\omega = 0$.

(b) [5 pts] Is there a smooth 1-form $\nu$ on $\mathbb{R}^3$ such that $d\nu = \omega$? Justify your answer.

(c) [5 pts] Let $B$ be the closed unit ball centered at the origin in $\mathbb{R}^3$. Calculate $\int_B \omega \wedge dy$.

(d) [5 pts] Let $K := \{(x, y, z) | z = 0, 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and let $i : K \to \mathbb{R}^3$ be the inclusion map. Calculate $\int_K i^*(\omega)$.

B02 [30 pts]: Explain why there is no embedding of the Mobius band in $\mathbb{R}^2$. Explain why there is an embedding of the Klein bottle in $\mathbb{R}^4$.

B03 [30 pts]: Let $n \geq 1$ be an integer. Let $M$ be a smooth $n$-manifold and let $f : M \to \mathbb{R}$ be smooth.

(a) [10 pts] Define a “regular value” of $f$.

(b) [10 pts] Describe the inverse image of a regular value of $f$. (No proof is required.)

(c) [10 pts] Show that it is possible for the inverse image of a critical (i.e. non-regular) value to be a submanifold of $M$.

B04 [20 pts]: Let $L$, $M$ and $N$ be the smooth vector fields on $\mathbb{R}^3$ defined by

$$L := y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad M := y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, \quad N := e^x y \frac{\partial}{\partial z}.$$

(a) [10 pts] Calculate $[-L, [M, L]] + [[L, M], L]$.

(b) [10 pts] Is it true that, for all $p \in \mathbb{R}^3$, the vector $[L, N]_p$ is a linear combination of the vectors $L_p$ and $N_p$?