Manifolds and Topology

Wednesday, 31 August 2016.

Please use separate bluebooks for parts A and B, and clearly label each bluebook, indicating its contents.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

All problems carry equal weight.
Part A.

1. Define what it means for two continuous paths $\alpha, \beta : [0, 1] \to X$, with the same start and end points, to be homotopic. Define the fundamental group $\pi_1(X, x)$ as a set.

2. Assume that $X$ is a space with a point $p$ and that $X$ deformation retracts onto $p$: there is a continuous map $H : [0, 1] \times X \to X$ such that
   
   - for all $x \in X$, $H(0, x) = x$,
   - for all $x \in X$, $H(1, x) = p$, and
   - for all $t \in [0, 1]$, $H(t, p) = p$.

   Prove that $\pi_1(X, p)$ has only one element. (Aside: the third condition on $H$ is not necessary, but makes the proof easier.)

3. Define the path concatenation operation $*$ on $\pi_1(X, x)$ and prove that it has a two-sided unit.

4. Use polar coordinates to construct an explicit covering map $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0, 0)\}$.

5. If $a, b, p$ are distinct points of $S^2$, determine the fundamental group $\pi_1(S^2 \setminus \{a, b\}, p)$ up to isomorphism.

6. Give a geometric description (with justification) of the universal cover of the space $S^1 \vee S^2$, obtained by gluing these two spaces $S^1$ and $S^2$ together at a point.

7. Give the statement (with all assumptions) of the Mayer–Vietoris long exact sequence for homology groups.

8. Consider the Möbius strip $M$ obtained by taking $[0, 1] \times [0, 1]$ and gluing together edges in opposing directions: $(0, s) \sim (1, 1 - s)$ for all $s \in [0, 1]$. Writing $i$ for the inclusion $\partial M \to M$, determine the map $i_* : H_1(\partial M) \to H_1(M)$.

9. For $n > 0$, define the degree of a continuous map $f : S^n \to S^n$.

10. Show that there exists no continuous map $f : S^2 \to S^2$ such that
    
    $$ f(f(x, y, z)) = (-x, -y, -z). $$
Part B.

1. Give an example (with proof) of a homeomorphism \( \mathbb{R} \to \mathbb{R} \) which is not a diffeomorphism.

2. Consider the function \( f(x, y, z) = (x^3 z, xy + z) \), from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \). At which points is \( f \) a submersion? Determine the regular values of \( f \).

3. Give an example of a smooth map \( f : \mathbb{R} \to \mathbb{R}^2 \) which is an immersion and not an embedding but where the image of \( f \) is still a submanifold of \( \mathbb{R}^2 \).

4. If \((x, y)\) are Cartesian coordinates and \((r, \theta)\) are polar coordinates on \( \mathbb{R}^2 \), express the vector field \( \frac{\partial}{\partial \theta} \) in Cartesian coordinates. Your answer should take the form \( P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \).

5. If \( \omega \) is the 1-form \( f(x, y, z) dy + g(x, y, z) dz \) on \( \mathbb{R}^3 \), give a formula for the 3-form \( \omega \wedge d\omega \), in terms of \( f \) and \( g \), in the simplest form you can.

6. Calculate the Lie bracket of the following vector fields on \( \mathbb{R}^3 \):

\[
\begin{align*}
\frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + (1 + 2xy) \frac{\partial}{\partial z} \\
y \frac{\partial}{\partial x} + (1 + xy) \frac{\partial}{\partial y} + (3y + 2xy^2) \frac{\partial}{\partial z}
\end{align*}
\]

Do there exist integral surfaces for this vector field? (In other words, for any point \( p \), does there exist a smooth surface \( S \) in \( \mathbb{R}^3 \) containing \( p \) whose tangent space at any point \( q \in S \) is spanned by these vector fields?)

7. State the Whitney embedding theorem on embeddings and immersions of \( m \)-dimensional smooth manifolds \( M \) into \( \mathbb{R}^k \). (If you know multiple versions, state the weak Whitney embedding/immersion theorems.)

8. Suppose \( \gamma \) is a simple closed curve in \( \mathbb{R}^2 \) which goes counterclockwise around the boundary of a region \( R \subset \mathbb{R}^2 \), and \( \vec{F} = N(x, y) \frac{\partial}{\partial x} + M(x, y) \frac{\partial}{\partial y} \) is a vector field with divergence \( \text{div}(\vec{F}) = (\frac{\partial}{\partial x} N + \frac{\partial}{\partial y} M) \). Show that the integral

\[
\int \int_R \text{div}(\vec{F}) \, dx dy dz
\]

can be calculated by a line integral of a vector field (sometimes called \( *\vec{F} \)) along \( \gamma \). How is this related to the divergence theorem (Gauss’ theorem) in \( \mathbb{R}^3 \)?

9. Let \( M = \mathbb{R}^3 \setminus S^2 \). For which values of \( k \) does there exist a smooth \( k \)-form \( \omega \) on \( M \) such that \( d\omega = 0 \), but \( \omega \neq d\tau \) for any \( (k-1) \)-form \( \tau \)?

10. Show that there cannot be an immersion \( S^1 \to \mathbb{R} \).