Complex Analysis
Ph.D Written Exam – Spring 2006

100 points are divided between 10 problems, 10 points each.
No books, no notes, no calculators.
There are two parts to this examination, part I and part II. Use separate blue books for these two parts; do NOT mix any part I with any part II in any blue book.

PART I.

Problem I–1. If $f(z)$ is holomorphic in a punctured disk $\{z : 0 < |z - a| < \varepsilon\}$, and has a non-removable singularity at $z = a$, show that $\exp f(z)$ has an essential singularity at $z = a$.

Problem I–2. Find the number of zeros of

$$f(z) = z^6 + 6z + 10$$

in the third quadrant.

Problem I–3. Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4} dx.$$

Problem I–4. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is entire and

$$|f(z)| \leq C |z|^\pi$$

for all $z$, with $|z| > 1$. Show that $f$ is a polynomial in $z$ of degree at most 3.

Problem I–5. Let $f$ be subharmonic on a connected open set $U \subseteq \mathbb{C}$, and let $D$ be a disk with $\overline{D} \subseteq U$. Assume $g$ is the harmonic function on $D$ which has continuous extension to $f$ on $\partial D$. Let

$$\tilde{f} = f \quad \text{on} \quad U \setminus D, \quad \tilde{f} = g \quad \text{on} \quad D.$$

Show that $\tilde{f}$ is subharmonic on $U$. 

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PART II.

Problem II–1. Suppose the points \( z_1, z_2, \ldots, z_n \) all lie on one side of a line drawn through the origin of the complex plane. Show that

\[
z_1 + z_2 + \ldots + z_n \neq 0, \quad \text{and} \quad \frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n} \neq 0.
\]

Problem II–2. Evaluate the integral

\[
\int_L \frac{z}{z} dz,
\]

where \( L \) is the boundary of the domain \( \{ z : 1 < |z| < 2, \Im z > 0 \} \) with the counterclockwise (positive) orientation.

Problem II–3. Expand

\[
f(z) = \ln \left( z + \sqrt{1 + z^2} \right)
\]

into a Taylor-Maclaurin series in \( \{|z| < 1\} \). Here the branches of \( \sqrt{\cdot} \) and \( \ln (\cdot) \) are chosen in such a way that \( f(0) = 0 \).

Problem II–4. Find an explicit conformal equivalence taking the region between two circles

\[
\left\{ z : \left| z - \frac{i}{2} \right| = 1 \right\} \quad \text{and} \quad \left\{ z : |z - i| = 1 \right\}
\]

onto the exterior of the unit circle.

Problem II–5. Let \( g(z) \) be an analytic function in a region \( \Omega \), and suppose the series

\[
\sum_{n=0}^{\infty} g^{(n)}(z)
\]

converges at a point \( z_0 \in \Omega \). Show that \( g(z) \) is an entire function, and this series converges uniformly on any circle \( \{|z| < R\} \).