Algebra Prelim Written Exam Fall 2017

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Show that a group of order $pq$ with primes $p < q$ and $q \neq 1 \mod p$ is necessarily abelian.

[2] Show that the quotient $(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \cdot (7, 11))$ is torsion-free.

[3] Let $S, T$ be diagonalizable operators on a finite-dimensional complex vector space $V$. Suppose that $ST = TS$. Show that there is a basis for $V$ consisting of simultaneous eigenvectors for both $S$ and $T$.

[4] Show that $x^5 + y^7 + 11$ is irreducible in $\mathbb{Z}[x, y]$.

[5] Describe all subfields of the field $\mathbb{F}_{64}$ with 64 elements.

[6] Let $k$ be a field. Show that the ideal $I$ generated by $x, y, z$ in $k[x, y, z]$ is maximal.

[7] Let $A, B, C$ be abelian groups, and let $0 \to A \to B \to C \to 0$ be an exact short sequence of homomorphisms of abelian groups, in the sense that the image of each map is the kernel of the next. Show that for every abelian group $X$

$$0 \to \text{Hom}(X, A) \to \text{Hom}(X, B) \to \text{Hom}(X, C)$$

is exact. By example, show that

$$0 \to \text{Hom}(X, A) \to \text{Hom}(X, B) \to \text{Hom}(X, C) \to 0$$

is not necessarily exact.