Decouplings and applications

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Let $\Xi$ be a collection of frequency points $\xi$ on some curved, compact manifold $S$ of diameter $\sim 1$ in $\mathbb{R}^n$ (e.g. the unit sphere $\mathbb{S}^{n-1}$)

Let $B_R = B(c, R)$ be a ball with radius $R \geq 1$. Let also $a_\xi \in \mathbb{C}$ be complex coefficients.

**Notation:** $e(z) = e^{2\pi i z}$.

**Question**

*How to estimate*

$$\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \|_{L^p(B_R)}$$
The case of tiny balls: spatial scale $\sim 1$

Since the Fourier transform $\sum_{\xi} a_{\xi} \delta_{\xi}$ of

$$\sum_{\xi} a_{\xi} e(\xi \cdot x)$$

is supported inside a ball of radius $\sim 1$, the Uncertainty Principle proves that the exponential sum is essentially constant on each ball $B(c, 1)$ of radius $\sim 1$

$$|\sum_{\xi} a_{\xi} e(\xi \cdot x)| \sim |\sum_{\xi} a_{\xi} e(\xi \cdot c)|, \ x \in B(c, 1).$$

The estimate via the triangle inequality

$$|\sum_{\xi \in \Xi} a_{\xi} e(\xi \cdot c)| \leq \sum_{\xi \in \Xi} |a_{\xi}|$$

is lossy, unless $|\Xi| = O(1)$. 
The case of medium size balls:
spatial scale $R \sim (\text{freq. separation})^{-1}$

A particular case of interest is when the points $\xi$ are separated by $\sim \frac{1}{R}$.

The $L^2$ estimate (in fact the equivalence)

$$
|B_R|^{-\frac{1}{2}} \left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^2(B_R)} \sim \| a_\xi \|_{L^2}
$$

is a rather immediate consequence of $L^2$ orthogonality.
The case of medium size balls continued

The value \( p = \frac{2(n+1)}{n-1} \) is special (Stein-Tomas exponent). If \( \xi \) are \( \frac{1}{R} \) separated on the paraboloid \( \mathbb{P}^{n-1} \), the sphere \( \mathbb{S}^{n-1} \) (or in general, on a hypersurface with nonzero Gaussian curvature) then

\[
\left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^{\frac{2(n+1)}{n-1}}(B_R)} \lesssim R^{\frac{n-1}{2}} \| a_\xi \|_2.
\]

This is a discrete equivalent reformulation of the Stein-Tomas restriction theorem (the sphere) and of the Strichartz estimate for the Schrödinger equation (the paraboloid).

The exponent \( \frac{n-1}{2} \) is sharp. The proof uses \( TT^* \) and the decay of Fourier transform of surface measure.
The case of medium size ball continued:

Let us continue to assume the points $\xi$ are separated by $\sim \frac{1}{R}$, on $S^{n-1}$ or $\mathbb{P}^{n-1}$. Another important $p$ is the restriction index

$$p = \frac{2n}{n-1}.$$ 

The estimate at $p = \frac{2n}{n-1}$

$$\left\| \sum_{\xi \in \Xi} a_{\xi} e(\xi \cdot x) \right\|_{L^p(B_R)} \lesssim \epsilon R^{n-1+\epsilon} \|a_{\xi}\|_{\infty}$$

is known in two dimensions, but open in higher dimensions. If proved, it would imply (among other things) that Kakeya sets in $\mathbb{R}^n$ have full Hausdorff dimension.

**Conclusion.** The medium ball case is fully understood (for all $p$) if $\|a_{\xi}\|_{l^2}$ is used on the right, and mostly open when $\|a_{\xi}\|_{l^\infty}$ is used.
The case of "large" balls: spatial scale $\sim (\text{freq. separation})^{-2}$

Let us now assume that the frequencies $\xi$ are $\frac{1}{R^{1/2}}$-separated. We will be interested in determining the sharp exponents $\alpha_{p,n}$ such that

$$\left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^p(B_R)} \lesssim R^{\alpha_{p,n}} \left\| a_\xi \right\|_{L^2}.$$ 

Note that the exponential sum has more space to oscillate than in the medium scale case, so it is expected that further cancellations would occur.

Of particular interest is to find the range $2 \leq p \leq p_c$ such that the following reverse Hölder holds

$$|B_R|^{-1/p} \left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^p(B_R)} \lesssim \left\| a_\xi \right\|_{L^2} \sim |B_R|^{-\frac{1}{2}} \left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^2(B_R)}$$

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The importance of the large spatial scale

Say we want to understand the size of
\[
\| \sum_{n=1}^{N} a_n e(n x_1 + n^2 x_2) \|_{L^p([0,1]^2, dx_1 dx_2)}.
\]

A change of variables reduces this to the integral
\[
\| \sum_{n=1}^{N} a_n e(x \cdot (\frac{n}{N}, \frac{n^2}{N^2})) \|_{L^p([0,N] \times [0,N^2], dx)},
\]
or, via periodicity to the integral on the large "ball"
\[
[0, N^2] \times [0, N^2]
\]
\[
\| \sum_{n=1}^{N} a_n e(x \cdot (\frac{n}{N}, \frac{n^2}{N^2})) \|_{L^p([0,N^2] \times [0,N^2], dx)}.
\]

Note that the points \((\frac{n}{N}, \frac{n^2}{N^2})\) \(\in \mathbb{P}^1\) are \(\frac{1}{N}\)-separated.
It turns out that

\[ |B_R|^{-1/p} \left\| \sum_{\xi \in \Xi} a_\xi e(\xi \cdot x) \right\|_{L^p(B_R)} \lesssim \epsilon R^\epsilon \|a_\xi\|_2^2 \]  

(1)

holds true for each \( 2 \leq p \leq \frac{2(n+1)}{n-1} \) and all \( \frac{1}{\sqrt{R}} \)-separated points \( \xi \) on \( S^{n-1}, \mathbb{P}^{n-1} \).

However, (1) is false if the large ball \( B_R \) is replaced with the medium size ball \( B_{\sqrt{R}} \).

(1) gives a Strichartz estimate in the (quasi)-periodic case of the (ir)rational torus. It will follow from the more general decoupling theory we will introduce next.
Let \((f_j)_{j=1}^N\) be \(N\) elements in a normed space \((X, \| \cdot \|_X)\). In this generality the triangle inequality

\[
\| \sum_{j=1}^N f_j \|_X \leq \sum_{j=1}^N \| f_j \|_X
\]

is the best estimate available for the norm of the sum of \(f_j\). When combined with the Cauchy–Schwarz inequality, it leads to

\[
\| \sum_{j=1}^N f_j \|_X \leq N^{1/2} \left( \sum_{j=1}^N \| f_j \|_X^2 \right)^{1/2}.
\]

Choosing \(X = L^1(\mathbb{R}^n)\) and positive functions \(f_j\) with equal \(L^1\) norms shows that the inequality above can be sharp.
However, if $X$ is a Hilbert space and if $f_j$ are pairwise orthogonal then we have a stronger inequality (in fact an equality)

$$\| \sum_{j=1}^{N} f_j \|_X \leq \left( \sum_{j=1}^{N} \| f_j \|_X^2 \right)^{1/2}.$$

An example which is ubiquitous in Fourier analysis is when $X = L^2(\mathbb{R}^n)$, and the $f_j$ are functions whose Fourier transforms are disjointly supported.

It is natural to ask whether there is an analogous phenomenon in $L^p(\mathbb{R}^n)$ when $p \neq 2$, in the absence of Hilbert space orthogonality.
Given a smooth function \( \psi : U \to \mathbb{R}^{n-d} \) we define the \( d \)-dimensional manifold in \( \mathbb{R}^n \)

\[
M = M^\psi = \{(\xi, \psi(\xi)) : \xi \in U\}.
\]

Examples of hypersurfaces (\( d = n - 1 \)) include the paraboloid

\[
\mathbb{P}^{n-1} = \{(\xi_1, \ldots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) : |\xi_i| < 1\},
\]

the hemispheres

\[
\mathbb{S}_\pm^{n-1} = \{(\xi, \pm \sqrt{1 - |\xi|^2}) : |\xi| < 1\},
\]

the truncated cone

\[
\mathbb{Cone}^{n-1} = \{(\xi, |\xi|) : 1 < |\xi| < 2\}.
\]

A "nice" curve (\( d = n - 1 \)) is the moment curve

\[
\Gamma_n = \{(t, t^2, \ldots, t^n) : t \in [0, 1]\}.
\]
Denote by $N_M(\delta)$ the $\delta$-neighborhood of $M$. Let $\Theta_\delta = \Theta_\delta(M)$ be a partition of $N_M(\delta)$ into box-like sets $\theta$ of thickness $\delta$. For each $\theta$ let $P_\theta F$ be the Fourier restriction of $F$ to $\theta$

$$P_\theta F = (\hat{F}1_\theta)^\vee \text{ or } \hat{P_\theta F} = \hat{F}1_\theta.$$ 

If $\hat{F}$ is supported inside $N_M(\delta)$ then

$$F = \sum_{\theta \in \Theta_\delta} P_\theta F$$

**Problem ($l^2$ decoupling for manifolds)***

*Find the range $2 \leq p \leq p_c$ such that the following $l^2$ decoupling holds for each $F$ with Fourier transform supported in $N_M(\delta)$*

$$\|F\|_{L^p(B(0,\delta^{-1}))} \lesssim \epsilon \delta^{-\epsilon} \left( \sum_{\theta \in \Theta_\delta} \|P_\theta F\|_{L^p(B(0,\delta^{-1}))}^2 \right)^{1/2}.$$
Bourgain’s observation (2011): To get from...

**Theorem (decoupling theorem for \( \mathcal{M} \))**

\[
\| F \|_{L^p(B(0,\delta^{-1}))} \lesssim \epsilon \, \delta^{-\epsilon} \left( \sum_{\theta} \| \mathcal{P}_\theta F \|_{L^p(B(0,\delta^{-1}))}^2 \right)^{1/2}.
\]

...to the exponential sum estimate (reverse Hölder)

**Theorem (Exponential sum estimate for \( \mathcal{M} \))**

For each \( \theta \in \Theta_\delta \) let \( \xi_\theta \in \theta \cap \mathcal{M} \) and \( a_\theta \in \mathbb{C} \). Then

\[
| B(0, \delta^{-1}) |^{-1/p} \left\| \sum_{\theta \in \Theta_\delta} a_\theta e(\xi_\theta \cdot x) \right\|_{L^p(B(0,\delta^{-1}))} \lesssim \epsilon \, \delta^{-\epsilon} \left( \sum_{\theta} | a_\theta |^2 \right)^{1/2}
\]

simply use (a smooth approximation of)

\[
F(\xi) = \sum_{\theta} a_\theta \delta_{\xi_\theta}
\]
The first decoupling was proved for the cone by T. Wolff (2000). The following is the first sharp (i.e. full range) decoupling (Bourgain covered the smaller range \(2 \leq p \leq \frac{2n}{n-1}\) in 2011).

**Theorem (Bourgain, D. 2014)**

For \(\delta \leq 1\), let \(\Theta_\delta\) be a partition of the \(\delta\)-neighborhood \(\mathcal{N}(\delta)\) of \(\mathbb{P}^{n-1}\) into essentially rectangular boxes of dimensions \(\delta^{1/2} \times \ldots \times \delta^{1/2} \times \delta\).

Let \(\hat{F}\) be supported in \(\mathcal{N}(\delta)\). If \(2 \leq p \leq \frac{2(n+1)}{n-1}\) we have

\[
\|F\|_{L^p(B(0,\delta^{-1}))} \lesssim \delta^{-\epsilon} \left( \sum_{\theta \in \Theta_\delta} \|P_\theta F\|_{L^p(B(0,\delta^{-1}))}^2 \right)^{1/2}.
\]

The critical exponent is the Stein–Tomas index \(p = \frac{2(n+1)}{n-1}\), and this is sharp. The same result holds for all hypersurfaces with nonzero Gaussian curvature.
In the following we will try to analyze the theorem for the parabola \( \mathbb{P}^1 \), at the critical index \( p_c = 6 \). We will call \( R = \delta^{-1} \).

**Theorem (Decoupling for the parabola)**

For \( \delta \leq 1 \), let \( \Theta_{R^{-1}} \) be a partition of the \( \frac{1}{R} \)-neighborhood \( \mathcal{N}(R^{-1}) \) of \( \mathbb{P}^1 \) into rectangles of dimensions \( R^{-1/2} \times R^{-1} \). We have

\[
\| F \|_{L^6([-R,R]^2)} \lesssim \epsilon R^\epsilon \left( \sum_{\theta \in \Theta_{R^{-1}}} \| \mathcal{P}_\theta F \|_{L^6([-R,R]^2)}^2 \right)^{1/2}.
\]

On the next slides we will explore some extremizers of this inequality using wave packets.
Wave packet decomposition on $[-R, R]^2$

For each $\theta \in \Theta_{R-1}$ we write

$$P_\theta F(x) = \sum_{T \in T_\theta} w_T W_T(x), \quad x \in [-R, R]^2$$

We assume all $(R^{1/2}, R)$-rectangles $T \in T_\theta$ are inside, and in fact tile $[-R, R]^2$. They are dual to $\theta$. Moreover

$$W_T(x) \approx 1_T(x)e(\xi_T \cdot x),$$

for some $\xi_T \in \theta \cap \mathbb{P}^1$. Also, $w_T \in \mathbb{C}$. Letting $\mathcal{T} = \bigcup_{\theta \in \Theta_{R-1}} T_\theta$ we call

$$F(x) = \sum_{T \in \mathcal{T}} w_T W_T(x)$$

the wave packet decomposition of $F$ on $[-R, R]^2$. 
Assume $\mathcal{T}$ contains $U \lesssim R^{1/2}$ rectangles for each direction. Assume also that there is a collection $S$ of roughly $U$ squares $S$ with unit side length inside $[-R, R]^2$, so that each $T$ intersects exactly one $S$. One way to realize this is to place all $S$ along a horizontal line segment at distance $\sim \frac{R}{U}$ from each other and to place a $T$ centered at each $S$, for each possible direction. Call $\mathcal{T}_S$ those $T \in \mathcal{T}$ that intersect $S$, and recall $\mathcal{T}_S$ are pairwise disjoint.

**Figure** $U = 2$ complete bushes
We choose all weights $w_T$ of magnitude roughly 1. Moreover, for each $S \in S$ with center $c_S$ we can make sure that the phases of $w_T$ are coordinated so that

$$| \sum_{T \in T_S} w_T W_T(c_S) | \gtrsim R^{1/2}.$$ 

This can be achieved due to our assumption that the collections $T_S$ are pairwise disjoint. The same inequality will in fact hold with $c_S$ replaced with $x$ satisfying $|x - c_S| \ll 1.$ Also, the contribution from those $T$ outside $T_S$ will be negligible. To conclude, we can make sure that

$$| \sum_{T \in T} w_T W_T(x) | \gtrsim R^{1/2}$$ 

holds on a set of area roughly $U.$ A simple computation shows that

$$\| F \|_{L^6} \gtrsim U^{1/6} R^{1/2}$$

and

$$\left( \sum_{\theta} \| \mathcal{P}_{\theta} F \|_{L^6}^2 \right)^{1/2} \sim U^{1/6} R^{1/2}.$$
Key elements of the proof

1. Bilinearization: the (linear) decoupling that we are trying to prove has an equivalent bilinear reformulation. This will allow us to take advantage of the bilinear Kakeya inequality.

2. Bilinear Kakeya inequality

3. $L^2$ local orthogonality

4. Parabolic rescaling and the fact that decouplings can be iterated

5. Multi-scale argument, induction on scales
Theorem (Bilinear Kakeya)

Consider two families $\mathbb{T}_1, \mathbb{T}_2$ consisting of rectangles $T$ in $\mathbb{R}^2$ having the following properties

(i) each $T$ has the short side of length $R^{1/2}$ and the long side of length equal to $R$ pointing in the direction of the unit vector $v_T$

(ii) $v_{T_1} \wedge v_{T_2} \geq \frac{1}{100}$ for each $T_i \in \mathbb{T}_i$.

We have the following inequality

$$\int_{\mathbb{R}^2} \sum_{T \in \mathbb{T}_1} 1_T \sum_{T \in \mathbb{T}_2} 1_T \lesssim R |\mathbb{T}_1| |\mathbb{T}_2|.$$ 

The implicit constant will not depend on $R, \mathbb{T}_i$. 

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Lemma (Iteration)

Let $\Theta$ be a collection consisting of boxes $\theta$. Assume each $\theta$ is partitioned into smaller boxes $\theta_1$. Call $\Theta_1$ the collection of all these smaller boxes. Assume that for each $\theta$

$$
\|P_{\theta} F\|_p \leq D_1 \left( \sum_{\theta_1 \subset \theta} \|P_{\theta_1} F\|_p^2 \right)^{1/2}.
$$

Assume also that

$$
\|F\|_p \leq D_2 \left( \sum_{\theta \in \Theta} \|P_{\theta} F\|_p^2 \right)^{1/2},
$$

whenever $\hat{F}$ is supported inside the union of all $\theta$. Then, for such an $F$ we also have

$$
\|F\|_p \leq D_1 D_2 \left( \sum_{\theta_1 \in \Theta_1} \|P_{\theta_1} F\|_p^2 \right)^{1/2}.
$$

The proof is an application of Minkowski’s inequality.
Fix $R = 2^{2s}$. For $1 \leq i \leq s$ we denote by $\Theta_i$ the partition of $N_{p^1}(R^{-2i+1})$ into nearly rectangular tubes $\theta_i$ of length $\sim R^{-2i}$.

Let $\text{Dec}(R)$ be the smallest constant such that for each $F : \mathbb{R}^2 \to \mathbb{C}$ with Fourier transform supported on $N(R^{-1})$ we have

$$\|F\|_{L^6([-R,R]^2)} \leq \text{Dec}(R) \left( \sum_{\theta_1 \in \Theta_1} \|P_{\theta_1} F\|_{L^6([-R,R]^2)}^2 \right)^{1/2}.$$ 

We need to prove that $\text{Dec}(R) \lesssim \varepsilon R^\varepsilon$. 
Fix two pairwise disjoint intervals $I_1, I_2 \subset [-1, 1]$, and denote by $\Theta_i(I_j)$ those $\theta_i \in \Theta_i$ that lie inside $\mathcal{N}_{I_j}(R^{-2^{-i+1}})$.

Let $\text{BilDec}(R)$ be the smallest constant such that for each $F_j : \mathbb{R}^2 \to \mathbb{C}$ with Fourier support inside $\mathcal{N}_{I_j}(R^{-1})$ we have

$$\| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \leq \text{BilDec}(R) \left( \sum_{\theta_1 \subset \Theta_1(I_1)} \| \mathcal{P}_{\theta_1} F_1 \|_{L^6([-R,R]^2)}^2 \right)^{1/4} \left( \sum_{\theta_1 \subset \Theta_1(I_2)} \| \mathcal{P}_{\theta_1} F_2 \|_{L^6([-R,R]^2)}^2 \right)^{1/4}.$$

One of the key ingredients of our proof is the fact that

$$\text{Dec}(R) \approx \text{BilDec}(R).$$
For the rest of the argument we fix $F_1, F_2$ with Fourier transforms supported in $\mathcal{N}_{I_1}(R^{-1}), \mathcal{N}_{I_2}(R^{-1})$ and write $F = F_1 + F_2$. Let us introduce the quantity $I_R(F_1, F_2)$ defined as follows

$$\frac{\|(F_1 F_2)^{1/2}\|_{L^6([-R,R]^2)}}{(\sum_{\theta_1 \subset \Theta_1(I_1)} \|\mathcal{P}_{\theta_1} F_1\|_{L^6([-R,R]^2)}^2)^{1/4} (\sum_{\theta_1 \subset \Theta_1(I_2)} \|\mathcal{P}_{\theta_1} F_2\|_{L^6([-R,R]^2)}^2)^{1/4}}.$$ 

The equivalence $\text{Dec}(R) \approx \text{BilDec}(R)$ allows us reformulate our goal in the form of the following inequality

$$I_R(F_1, F_2) \lesssim \epsilon R^\epsilon.$$ 

We will seek various estimates for $I_R(F_1, F_2)$, via wave packet decompositions at multiple scales and repeated pigeonholing. There will be many parameters at each scale, that reflect the local properties of the two functions.
Let us denote $[-R, R]^2$ by $S_1$ and let $S_2 = S_2(S_1)$ be the partition of $S_1$ into squares $S_2$ of side length $R^{1/2}$. For $1 \leq j \leq 2$ we use a wave packet representation of $F_j$ at scale $R$ on $[-R, R]^2$

$$F_j = \sum_{T_1 \in T_j} w_{T_1} W_{T_1}.$$ 

To recap, each $T_1$ is a rectangle with dimensions $R, R^{1/2}$, which is roughly dual to some $\theta_1 \in \Theta_1(I_j)$. Also, $W_{T_1}$ has Fourier transform supported inside $\theta_1(T_1)$ and

$$|W_{T_1}(x)| \approx 1_{T_1}(x).$$
Proposition (Pigeonholing at scale $R$)

There are positive integers $M_1, N_1, U_1, V_1, \beta_1, \gamma_1$ and real numbers $g_1, h_1 > 0$, a collection $S_2^* \subset S_2$ of $R^{1/2}$-squares $S_2$ and families of rectangles $T_1^* \subset T_1$, $T_2^* \subset T_2$ such that

- (uniform weight) For each $T_1 \in T_1^*$ we have $|w_{T_1}| \sim g_1$ and for each $T_1 \in T_2^*$ we have $|w_{T_1}| \sim h_1$.

- (uniform number of rectangles per direction) There is a family of $\sim M_1$ tubes in $\Theta_1(I_1)$ such that all rectangles $T_1 \in T_1^*$ are dual to some $\theta_1$ in this family, with $\sim U_1$ rectangles for each such $\theta_1$. In particular, the size of $T_1^*$ is $\sim M_1 U_1$.

Similarly, there is a family of $\sim N_1$ tubes in $\Theta_1(I_2)$ such that all rectangles $T_1 \in T_2^*, S_1$ are dual to some $\theta_1$ in this family, with $\sim V_1$ rectangles for each such $\theta_1$. In particular, the size of $T_2^*, S_1$ is $\sim N_1 V_1$. 
Proposition (Pigeonholing at scale $R$, continued)

- (uniform number of rectangles per square) Each $S_2 \in S_2^*$ intersects $\sim \frac{M_1}{\beta_1}$ rectangles from $T_1^*$ and $\sim \frac{N_1}{\gamma_1}$ rectangles from $T_2^*$.

Moreover

$$\| (F_1 F_2)^{1/2} \|_{L^6(S_1)} \lesssim \| (F_1^{(1)} F_2^{(1)})^{1/2} \|_{L^6(\bigcup_{S_2 \in S_2^*} S_2)} ,$$

where for $1 \leq j \leq 2$

$$F_j^{(1)} = \sum_{T_1 \in T_j^*} w_{T_1} W_{T_1} .$$
Recall we aim to prove an $O(R^\epsilon)$ estimate for

$$\frac{\| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)}}{(\sum_{\theta_1 \subset \Theta_1(l_1)} \| \mathcal{P}_{\theta_1} F_1 \|_{L^6([-R,R]^2)}^2)^{1/4} (\sum_{\theta_1 \subset \Theta_1(l_2)} \| \mathcal{P}_{\theta_1} F_2 \|_{L^6([-R,R]^2)}^2)^{1/4}}.$$

After just one round of pigeonholing we have a satisfactory (in fact definitive) estimate for the denominator.

**Proposition**

We have

$$\left( \sum_{\theta_1 \subset \Theta_1(l_1)} \| \mathcal{P}_{\theta_1} F_1 \|_{L^6([-R,R]^2)}^2 \right)^{1/4} \left( \sum_{\theta_1 \subset \Theta_1(l_2)} \| \mathcal{P}_{\theta_1} F_2 \|_{L^6([-R,R]^2)}^2 \right)^{1/4} \gtrsim R^{1/4} (M_1 N_1)^{1/4} (g_1 h_1)^{1/2} (U_1 V_1)^{1/12}.$$
On the other hand, the estimate for

\[ \| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \]

is still difficult at this point, since each \( x \in [-R, R]^2 \) receives contribution from many tubes.

Recall that the first round of pigeonholing produced a collection \( S^*_2 \) of significant squares \( S_2 \) with side length \( R^{1/2} \), where a big fraction of \( \| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \) is concentrated.

We would like to repeat the wave packet decomposition on each \( S_2 \in S_2 \). Then we will further repeat this until we reach scale \( \sim 1 \).

But first we need to estimate the size of \( S^*_2 \).
Proposition (Counting squares via bilinear Kakeya)

We have

\[ |S^*_2| \lesssim \beta_1 \gamma_1 U_1 V_1 \]

Proof.

We apply the bilinear Kakeya inequality. More precisely we have the following inequality for the fat rectangles $10T_1$

\[
\int_{\mathbb{R}^2} \left( \sum_{T_1 \in T_1^*} 1_{10 T_1} \right) \left( \sum_{T_1 \in T_2^*} 1_{10 T_1} \right) \lesssim R M_1 N_1 U_1 V_1.
\]

Recall also that each $S_2 \in S_2^*$ is contained in $\sim \frac{M_1}{\beta_1}$ fat rectangles $10T_1$ with $T_1 \in T_1^*$ and in $\sim \frac{N_1}{\gamma_1}$ fat rectangles $10T_1$ with $T_1 \in T_2^*$. With this observation, the bound on $|S^*_2|$ follows immediately.
On each $S_2$ write

$$F_j = \sum_{T_2 \in \mathcal{T}_j, S_2} w_{T_2} W_{T_2},$$

Each $T_2$ is a rectangle with dimensions $R^{1/2}, R^{1/4}$, which is roughly dual to some $\theta_2 \in \Theta_2(I_j)$. Also, $W_{T_1}$ has Fourier transform supported inside $\theta_2$ and

$$|W_{T_2}(x)| \approx 1_{T_2}(x).$$

Wave packets at two scales
Proposition (Pigeonholing at scale $R^{1/2}$)

We may refine the collection $S_2^*$ to get a smaller collection $S_2^{**}$ of squares of side length $R^{1/2}$, so that for each $S_2 \in S_2^{**}$ the following hold.

There are positive integers (independent of $S_2$) $M_2, N_2, U_2, V_2, \beta_2, \gamma_2$ and real numbers $g_2, h_2 > 0$, a collection $S_3^*(S_2)$ of $R^{1/4}$-squares $S_3$ inside $S_2$ and two families of rectangles $T_1^{*}, S_2 \subset T_1, S_2, T_2^{*}, S_2 \subset T_2, S_2$ such that

- **(uniform weight)** For each $T_2 \in T_1^{*}, S_2$ we have $|w_{T_2}| \sim g_2$ and for each $T_2 \in T_2^{*}, S_2$ we have $|w_{T_2}| \sim h_2$.

- **(uniform number of rectangles per direction)** There is a family of $\sim M_2$ tubes $\theta_2$ in $\Theta_2(I_1)$ such that each $T_2 \in T_1^{*}, S_2$ is dual to such a $\theta_2$, with $\sim U_2$ rectangles for each such $\theta_2$. In particular, the size of $T_1^{*}, S_2$ is $\sim M_2 U_2$. family of $\sim N_2$ tubes $\theta_2 \in \Theta_2(I_2)$ such that each $T_2 \in T_2^{*}, S_2$ is dual to such a $\theta_2$, with $\sim V_2$ rectangles for each such $\theta_2$. In particular, the size of $T_2^{*}, S_2$ is $\sim N_2 V_2$. 

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Decouplings and applications
Proposition (Pigeonholing at scale $R^{1/2}$, continued)

• (uniform number of rectangles per square) Each $S_3 \in S_3^*(S_2)$ intersects $\sim \frac{M_2}{\beta_2}$ rectangles from $T_{1, S_2}^*$ and $\sim \frac{N_2}{\gamma_2}$ rectangles from $T_{2, S_2}^*$.

Moreover

$$\| (F_1 F_2)^{1/2} \|_{L^6(S_1)} \lesssim \| (F_1^{(2)} F_2^{(2)})^{1/2} \|_{L^6(\cup_{S_3 \in S_3^*} S_3)},$$

(2)

where for $1 \leq j \leq 2$

$$F_j^{(2)} = \sum_{S_2 \in S_2^{**}} \sum_{T_2 \in T_j^*} w_{T_2} W_{T_2}$$

and

$$S_3^* = \{ S_3 \in S_3^*(S_2) : S_2 \in S_2^{**} \}.$$
To find the relation between the parameters $M_1, N_1, \ldots$ and $M_2, N_2$ we use local $L^2$ orthogonality.

**Proposition (Local orthogonality)**

We have

\[ U_2 g_2^2 R_4^3 M_2 \lesssim R g_1^2 \frac{M_1}{\beta_1}, \]

\[ V_2 h_2^2 R_4^3 N_2 \lesssim R h_1^2 \frac{N_1}{\gamma_1}. \]

Recall that on each $S_2 \in S^{**}_2$ we have two ways of representing $F^{(1)}_1$, corresponding to the scale $R$ and $R^{1/2}$, namely

\[ F^{(1)}_1(x) \approx \sum_{T_1 \in T^*_1 \atop T_1 \cap S_2 \neq \emptyset} w_{T_1} W_{T_1}(x), \quad x \in S_2 \]

and

\[ F^{(1)}_1(x) \approx \sum_{T_2 \in T_1, S_2} w_{T_2} W_{T_2}(x), \quad x \in S_2. \]
We can repeat the previous argument to count the significant squares \( S_3 \) inside each significant square \( S_2 \).

**Proposition (Counting squares via bilinear Kakeya)**

We have

\[
|S_3^*| \lesssim \beta_2 \gamma_2 U_2 V_2 |S_2^*| \\
\lesssim (\beta_2 \gamma_2 U_2 V_2)(\beta_1 \gamma_1 U_1 V_1)
\]
We have so far done wave packet decompositions at two scales: $R$ and $R^{1/2}$. Recall that $R = 2^{2^s}$. Iterate this $s$ times until we reach scale 2. Both $F_1$ and $F_2$ will be concentrated on squares $S_{s+1}$ of side length $\sim 1$.

$$\| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \lesssim \| (F_{1}^{(s)} F_{2}^{(s)})^{1/2} \|_{L^6(\bigcup_{S_{s+1} \in S^{*}_{s+1}} S_{s+1})},$$

**Key achievement:** There will be only $O(1)$ wave packets contributing to both $F_1$ and $F_2$:

$$F_{1}^{(s)} \approx \sum_{T_s} g_s W_{T_s},$$

$$F_{2}^{(s)} \approx \sum_{T_s} h_s W_{T_s}$$

where each $T_s$ is now a unit square. At this tiny scale we may write

$$|F_{1}^{s}| \sim |g_s|, |F_{2}^{s}| \sim |h_s| \text{ on } \bigcup_{S_{s+1} \in S^{*}_{s+1}} S_{s+1}.$$
Combining

$$\| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \lesssim \| (F_1^{(s)} F_2^{(s)})^{1/2} \|_{L^6(\cup_{s+1 \in S_{s+1}^*} S_{s+1})},$$

with

$$|F_1^s| \sim |g_s|, |F_2^s| \sim |h_s| \text{ on } \cup_{s+1 \in S_{s+1}^*} S_{s+1}$$

and (via $s$ applications of bilinear Kakeya)

$$|S_{s+1}^*| \lesssim \prod_{k=1}^{s} (\beta_k \gamma_k U_k V_k)$$

leads to a good estimate

$$\| (F_1 F_2)^{1/2} \|_{L^6([-R,R]^2)} \lesssim (g_s h_s)^{1/2} \prod_{k=1}^{s} (\beta_k \gamma_k U_k V_k)^{1/6}$$
Conclusion: The three estimates

\[
\|(F_1 F_2)^{1/2}\|_{L^6([-R,R]^2)} \lesssim (g_s h_s)^{1/2} \prod_{k=1}^{s}(\beta_k \gamma_k U_k V_k)^{1/6}
\]

\[
(\sum_{\theta_1 \subset \Theta_1(l)} \|P_{\theta_1} F_1\|_{L^6([-R,R]^2)}^2)^{1/4} (\sum_{\theta_1 \subset \Theta_1(l)} \|P_{\theta_1} F_2\|_{L^6([-R,R]^2)}^2)^{1/4}
\]

\[
\gtrsim R^{1/4} (M_1 N_1)^{1/4} (g_1 h_1)^{1/2} (U_1 V_1)^{1/12}
\]

and local \(L^2\) orthogonality at each scale

\[
\frac{g_k h_k}{g_{k-1} h_{k-1}} \lesssim R^{2-k} \frac{1}{(\beta_{k-1} \gamma_{k-1})^{1/2}} \frac{1}{(U_k V_k)^{1/2}} \left(\frac{M_{k-1} N_{k-1}}{M_k N_k}\right)^{1/2}
\]

lead to the desired estimate

\[
\text{Dec}(R) \lesssim R^\epsilon.
\]
We have so far established the optimal decoupling theory for the following manifolds

- Hypersurfaces in $\mathbb{R}^n$ with nonzero Gaussian curvature. **Applications:** Optimal Strichartz estimates for Shrödinger equation on both rational and irrational tori in all dimensions, improved $L^p$ estimates for the eigenfunctions of the Laplacian on the torus, Schrödinger maximal function (Du, Guth, Li)

- The cone (zero Gaussian curvature) in all dimensions. **Applications:** progress on Sogge’s “local smoothing conjecture for the wave equation” ; the boundedness properties of the Bergman projection in tube domains over full light cones, averages of Riemann zeta on short intervals (Bourgain-Watt)

- Two dimensional surfaces in $\mathbb{R}^4$. **Application:** Bourgain used this to improve the estimate in the Lindelöf hypothesis for the growth of Riemann zeta on the critical line
• (with Larry, too) Curves with torsion in all dimensions, such as

\[ \Gamma_n = \{ (t, t^2, \ldots, t^n) : t \in [0, 1] \}. \]

**Application:** Vinogradov’s Mean Value Theorem.

• Two dimensional manifolds in \( \mathbb{R}^n \) and beyond, work by Bourgain-Guo-D. and more recently by Guo-Zhang, completely solving the higher dimensional analogs of the Vinogradov systems.