Classification of solvable Lie algebras - new approaches and developments

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Outline

1. Classification of Lie algebras in general
2. Classification of solvable Lie algebras
3. Classification for a particular sequence of nilradicals
4. Conclusions
What is known in general about the classification of finite–dimensional Lie algebras over the fields of complex and real numbers?
If $\mathfrak{g}$ is decomposable into a direct sum of ideals, it should be explicitly decomposed into components that are further indecomposable

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k.$$  \hspace{1cm} (1)

An algorithm realizing the decomposition (1) exists\(^1\).

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\(^1\)Rand D, Winternitz P and Zassenhaus H 1988 *Linear algebra and its applications* **109** 197–246
Levi decomposition

Let $\mathfrak{g}$ denote an (indecomposable) Lie algebra. A fundamental theorem due to E. E. Levi\(^2\) tells us that any Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}, \quad [\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}, \quad [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, \quad [\mathfrak{l}, \mathfrak{r}] \subset \mathfrak{r},$$

(2)

where $\mathfrak{l}$ is a semisimple subalgebra and $\mathfrak{r}$ is the radical of $\mathfrak{g}$, i.e. its maximal solvable ideal.

An algorithm realizing the decomposition (2) exists\(^3\).

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\(^3\)Rand D, Winternitz P and Zassenhaus H 1988 *Linear algebra and its applications* **109** 197–246
Semisimple Lie algebras over the field of complex numbers $\mathbb{C}$ have been completely classified by E. Cartan$^4$, over the field of real numbers $\mathbb{R}$ by F. Gantmacher$^5$.

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$^4$Cartan E 1894 *Sur la structure des groupes de transformations finis et continus* (Paris: Thesis, Nony)

We note that by virtue of Jacobi identities $\tau$ is a representation space for $\mathfrak{l}$ and that $\mathfrak{l}$ is isomorphic to a subalgebra of the algebra of all derivations of $\tau$.

These observations put a rather stringent compatibility conditions on possible pairs of $\mathfrak{l}, \tau$ and can be employed in the construction of Levi decomposable algebras out of the classifications of semisimple and solvable ones. E.g. many solvable algebras do not have any semisimple subalgebra of derivations and hence cannot appear as a radical in a nontrivial Levi decomposition.

**BUT** not all solvable Lie algebras are known.
There are two ways of proceeding in the classification of solvable Lie algebras: by dimension, or by structure.

The dimensional approach for real Lie algebras:

- dimension 4: Kruchkovich GI 1954, Usp. Mat. Nauk 9 59
The classification of low–dimensional Lie algebras over \( \mathbb{C} \) was started earlier by S. Lie himself (Lie S and Engel F 1893 *Theorie der Transformationsgruppen III*, Leipzig: B.G. Teubner).

Some incomplete classifications are known for solvable Lie algebras in dimension 7 and nilpotent algebras up to dimension 8 (M.P. Gong, Gr. Tsagas, A.R. Parry).

It seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras \( \mathfrak{g} \) beyond \( \dim \mathfrak{g} = 6 \). It is however possible to proceed by structure, i.e. to classify solvable Lie algebras with the nilradical of a given type.


Basic concepts and notation

Three series of subalgebras – characteristic series of \( g \):

- **derived series** \( g = g^{(0)} \supseteq \ldots \supseteq g^{(k)} \supseteq \ldots \) defined

  \[ g^{(k)} = [g^{(k-1)}, g^{(k-1)}], \quad g^{(0)} = g. \]

  If \( \exists k \in \mathbb{N} \) such that \( g^{(k)} = 0 \), then \( g \) is solvable.

- **lower central series** \( g = g^1 \supseteq \ldots \supseteq g^k \supseteq \ldots \) defined

  \[ g^k = [g^{k-1}, g], \quad g^1 = g. \]

  If \( \exists k \in \mathbb{N} \) such that \( g^k = 0 \), then \( g \) nilpotent. The largest value of \( K \) s.t. \( g^K \neq 0 \) is the degree of nilpotency.

- **upper central series** \( z_1 \subseteq \ldots \subseteq z_k \subseteq \ldots \subseteq g \) where \( z_1 \) is the center of \( g \), \( z_1 = C(g) = \{ x \in g | [x, y] = 0, \forall y \in g \} \) and \( z_k \) are the higher centers defined recursively through

  \[ z_{k+1}/z_k = C(g/z_k). \]
Any solvable Lie algebra $\mathfrak{s}$ has a uniquely defined nilradical $NR(\mathfrak{s})$, i.e. maximal nilpotent ideal. Its dimension satisfies

$$\dim NR(\mathfrak{s}) \geq \frac{1}{2} \dim \mathfrak{s}. \quad (3)$$

The derived algebra of a solvable Lie algebra $\mathfrak{s}$ is contained in the nilradical, i.e.

$$[\mathfrak{s}, \mathfrak{s}] \subseteq NR(\mathfrak{s}). \quad (4)$$

The centralizer $\mathfrak{g}_\mathfrak{h}$ of a given subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.

$$\mathfrak{g}_\mathfrak{h} = \{ x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{h} \}. \quad (5)$$
A derivation $D$ of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D : \mathfrak{g} \to \mathfrak{g}$$

such that for any pair $x, y$ of elements of $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \quad (6)$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \text{ad}_z, \quad \text{i.e. } D(x) = [z, x], \; \forall x \in G,$$

the derivation is inner, any other one is outer.
An automorphism $\Phi$ of $g$ is a regular linear map

$$\Phi : g \to g$$

such that for any pair $x, y$ of elements of $g$

$$\Phi([x, y]) = [\Phi(x), \Phi(y)]. \quad (7)$$

Ideals in the characteristic series as well as their centralizers are invariant with respect to all derivations and automorphisms, i.e. belong among characteristic ideals.
Classification of solvable Lie algebras with the given nilradical

We assume that the nilradical $\mathfrak{n}$, $\dim \mathfrak{n} = n$ is known. That is, in some basis $(e_1, \ldots, e_n)$ we know the Lie brackets

$$[e_i, e_j] = N_{ij}^k e_k.$$  \hfill (8)

We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_f$ to the basis $(e_1, \ldots, e_n)$ which together will form a basis of $\mathfrak{s}$. It follows from (4) that

$$[f_a, e_i] = (A_a)_i^j e_j, \ 1 \leq a \leq f, \ 1 \leq j \leq n,$$

$$[f_a, f_b] = \gamma_{ab}^i e_i, \ 1 \leq a, b \leq f.$$  \hfill (9)
We have

- Jacobi identities between $(f_a, e_i, e_j) \implies$ linear homogeneous equations for the matrix elements of $A_i$.
- Jacobi identities between $(f_a, f_b, e_i) \implies$ linear inhomogeneous equations for $\gamma_{ab}^i$ in terms of the commutators of $A_a$ and $A_b$.
- Jacobi identities between $(f_a, f_b, f_c) \implies$ bilinear compatibility conditions on $\gamma_{ab}^i$ and $A_a$.

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$, no nontrivial linear combination of $A_a$ can be a nilpotent matrix, i.e. they are linearly nil–independent.
Let us consider the adjoint representation of $s$ restricted to the nilradical $\mathfrak{n}$. Then $\text{ad}(f_a)|_\mathfrak{n}$ is a derivation of $\mathfrak{n}$. In other words, finding all sets of matrices $A_a$ in (9) is equivalent to finding all sets of outer nil–independent derivations of $\mathfrak{n}$

$$D_1 = \text{ad}(f_a)|_\mathfrak{n}, \ldots, D_f = \text{ad}(f_f)|_\mathfrak{n},$$

(10)
such that $[D_a, D_b]$ are inner derivations.

$\gamma_{ab}^i$ are then determined up to elements in the center $C(\mathfrak{n})$ of $\mathfrak{n}$ by $[D_a, D_b] = \gamma_{ab}^i \text{ad}(e_i)|_\mathfrak{n}$, i.e. the knowledge of all sets of such derivations almost amounts to the knowledge of all solvable Lie algebras with the given nilradical $\mathfrak{n}$. 
If we

1. **add any inner derivation** to $D_a$, i.e. we consider outer derivations modulo inner derivations,

2. **perform a change of basis in $\mathfrak{n}$** such that the Lie brackets (8) are not changed, i.e. we consider only conjugacy classes of sets of outer derivations (modulo inner derivations)

3. **change the basis in the space $\text{span}\{D_1, \ldots, D_f\}$,**

the resulting Lie algebra is **isomorphic** to the original one.
Suitable basis of \( n \) to begin with

Starting with any complement \( m_1 \) of \( n^2 \) in \( n \) one can construct a sequence of subspaces \( m_j \) such that

\[
n = m_K + m_{K-1} + \ldots + m_1 \tag{11}
\]

where

\[
n^j = m_j + n^{j+1}, \quad m_j \subset [m_{j-1}, m_1]. \tag{12}
\]

By construction of these subspaces, any derivation (automorphism) is determined once its action on \( m_1 \) is known. We shall assume that we work in a basis of \( n \) which respects the decomposition (11).
Because \( n^j = m_K + \ldots + m_j \), any derivation now takes a block triangular form

\[
D = \begin{pmatrix}
D_{m_Km_K} & \cdots & D_{m_Km_2} & D_{m_Km_1} \\
\vdots & \ddots & \vdots & \vdots \\
D_{m_2m_2} & \cdots & D_{m_2m_1} \\
D_{m_1m_1} & & & \\
\end{pmatrix}.
\] (13)

where the elements of \( D_{mjm_k}, k \leq j = 2, \ldots, K \) are linear functions of elements in the last column blocks \( D_{m_1m_1}, \ldots, D_{mj-k+1m_1} \).
Now one can easily establish that:

- Any inner derivation has vanishing diagonal blocks.
- A derivation $D$ is nilpotent if and only if $D_{m_1m_1}$ is nilpotent.
- Derivations $D_1, \ldots, D_f$ are linearly nilindependent if and only if $(D_1)_{m_1m_1}, \ldots, (D_f)_{m_1m_1}$ are linearly independent.
- If all pairwise commutators of the derivations $D_1, \ldots, D_f$ are inner derivations then necessarily $(D_1)_{m_1m_1}, \ldots, (D_f)_{m_1m_1}$ must pairwise commute.
Estimate on maximal value of $f$

$f = \dim s - \dim n$ is bounded by the maximal number of commuting $m_1 \times m_1$ matrices. i.e. satisfies

$$f \leq \dim n - \dim n^2.$$  \hfill (14)

This estimate is different from the one derived by Mubarakzyanov

$$f \leq \dim n$$ \hfill (15)

and improved by him to

$$f \leq \dim n - \dim C(s).$$ \hfill (16)
Comparison of the estimates

There are at least two advantages to the estimate (14) over (16):

- the bound (14) is in most cases more restrictive than (16),
- and it doesn’t depend on the knowledge of the structure of the whole solvable Lie algebra $s$, contrary to the bound (16).

The bound (14) is saturated for many classes of nilpotent Lie algebras whose solvable extensions were previously investigated – e.g. Abelian, naturally graded filiform $n_{n,1}$, $Q_n$, a decomposable central extension of $n_{n,1}$, and of nilpotent triangular matrices.
On the other hand, it is obvious that even the improved bound (14) cannot give a precise estimate on the maximal dimension of a solvable extension in all cases. In particular, we have always \( \dim \mathfrak{n} - \dim \mathfrak{n}^2 \geq 2 \), i.e. characteristically nilpotent Lie algebras cannot be easily detected using Eq. (14).

Similarly, the bound (14) is not saturated in the case of Heisenberg nilradicals \( \mathfrak{h} \) where the maximal number of non-nilpotent elements is in fact equal to \( \frac{\dim \mathfrak{h} + 1}{2} < \dim \mathfrak{h} - 1 \).
The estimate (14) also allows us to construct a lower bound on dimension of the nilradical of a given solvable Lie algebra \( \mathfrak{s} \). We have

\[
\dim \mathfrak{s} + \dim \mathfrak{n}^2 \leq 2 \dim \mathfrak{n}
\]

and \( \mathfrak{s}^{(2)} = (\mathfrak{s}^2)^2 \subset \mathfrak{n}^2 \) because \( \mathfrak{s}^2 \subset \mathfrak{n} \). Altogether, we find

\[
\dim \mathfrak{n} \geq \frac{1}{2} \left( \dim \mathfrak{s} + \dim \mathfrak{s}^{(2)} \right). \tag{17}
\]

In our experience, this estimate is often less accurate than the trivial estimate \( \dim \mathfrak{n} \geq \dim \mathfrak{s}^2 \). Nevertheless, the bound (17) can be useful in some particular cases.
Classification for a particular sequence of nilradicals

Our nilradical $\mathfrak{n}_{n,3}$

$$
\begin{align*}
[e_2, e_n] &= e_1, \\
[e_3, e_{n-1}] &= e_1, \\
[e_4, e_n] &= e_2, \\
[e_k, e_n] &= e_{k-1}, \quad 5 \leq k \leq n-2, \\
[e_{n-1}, e_n] &= -e_3.
\end{align*}
$$

(18)

It has the following complete flag of invariant ideals

$$
0 \subset \mathfrak{n}^{n-3} \subset \mathfrak{n}^{n-4} \subset \mathfrak{z}_2 \subset \mathfrak{z}_3 \cap \mathfrak{n}^2 \subset \ldots \subset \mathfrak{z}_{n-5} \cap \mathfrak{n}^2 \subset \mathfrak{n}^2 \\
\subset (\mathfrak{z}_2)_n \subset (\mathfrak{n}^{n-4})_n \subset \mathfrak{n}
$$

(19)

and a subalgebra isomorphic to $\mathfrak{n}_{n-2,1}$ expressed as

$$
\tilde{\mathfrak{n}}_{n-2,1} = \text{span}\{e_1, e_2, e_4, \ldots, e_{n-2}, e_n\}.
$$

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$^6$L. Šnobl and D. Karásek 2010 *Linear algebra Appl.* 432 1836–1850
What is important for us is that the solvable extensions of $\mathfrak{n}_{n-2,1}$ were fully investigated in Šnobl L and Winternitz P 2005, A class of solvable Lie algebras and their Casimir invariants, *J. Phys. A* 38 2687. At the same time, the group of automorphisms of $\mathfrak{n}_{n-2,1}$ is almost the same as the one induced on $\tilde{\mathfrak{n}}_{n-2,1}$ by automorphisms of $\mathfrak{n}_{n,3}$. More precisely, locally they are identical, globally they differ by one reflection allowed in $\mathfrak{n}_{n-2,1}$ but not in $\tilde{\mathfrak{n}}_{n-2,1}$. 
Our current approach is as follows

1. Check which conjugacy classes of elements in $\text{Der}(n_{n,3})/\text{Inn}(n_{n,3})$ can be represented by derivations which preserve $\tilde{n}_{n-2,1}$.

2. For these find all solvable extensions of $\tilde{n}_{n-2,1}$ and extend them to solvable extensions of $n_{n,3}$. In this process some new parameters may arise, i.e. the extension is not necessarily unique.

3. Consider the classes of derivations which don’t have any representative preserving $\tilde{n}_{n-2,1}$ and construct the corresponding solvable extensions.
Some details

We choose the subspaces $m_i$ in the form

$$m_1 = \text{span}\{e_{n-2}, e_{n-1}, e_n\}, \quad m_2 = \text{span}\{e_3, e_{n-3}\},$$

$$m_i = \text{span}\{e_{n-1-i}\}, \quad i = 3, \ldots, n-5,$$

$$m_{n-4} = \text{span}\{e_2\}, \quad m_{n-3} = n^{n-3} = \text{span}\{e_1\}.$$

The most general derivation is specified by

$$D(e_{n-2}) = (2c_{n-1} + (5 - n)d_n)e_{n-2} + \sum_{k=4}^{n-3} b_k e_k + b_2 e_2 + b_1 e_1,$$

$$D(e_{n-1}) = c_{n-1} e_{n-1} + d_{n-1} e_4 + \sum_{k=1}^{3} c_k e_k, \quad (20)$$

$$D(e_n) = \sum_{k=1}^{n} d_k e_k.$$
In the $2n$–dimensional algebra of derivations $\text{Der}(n_{n,3})$ we have an $(n - 1)$–dimensional ideal of inner derivations $\text{Inn}(n_{n,3})$ of the form

\begin{align}
D(e_{n-2}) &= -c_3 e_{n-3}, \\
D(e_{n-1}) &= c_3 e_3 + c_1 e_1, \\
D(e_n) &= \sum_{k=1}^{n-3} d_k e_k.
\end{align}  \tag{21}
The elements of $\text{Der}(n_{n,3})/\text{Inn}(n_{n,3})$ can be uniquely represented by outer derivations of the form

\[
D(e_{n-2}) = (2c_{n-1} + (5 - n)d_n)e_{n-2} + \sum_{k=4}^{n-4} b_k e_k + b_2 e_2 + b_1 e_1,
\]
\[
D(e_{n-1}) = c_{n-1}e_{n-1} + d_{n-1}e_4 + c_3 e_3 + c_2 e_2,
\]
\[
D(e_n) = d_ne_n + d_{n-1}e_{n-1} + d_{n-2}e_{n-2}.
\]

(22)

The derivation of the form (22) leaves $\tilde{n}_{n-2,1}$ invariant if and only if $d_{n-1} = 0$. We conjugate a given derivation $D$ by the automorphism defined by

\[
e_{n-2} \rightarrow e_{n-2}, \quad e_{n-1} \rightarrow e_{n-1} + \frac{d_{n-1}}{d_n - c_{n-1}} e_4, \quad e_n \rightarrow e_n + \frac{d_{n-1}}{d_n - c_{n-1}} e_{n-1}
\]

whenever possible, i.e. when $d_n \neq c_{n-1}$. 
Now we have $\hat{d}_{n-1} = 0$, i.e. $D_\Phi \equiv \hat{D}$ leaves $\tilde{n}_{n-2,1}$ invariant and we can proceed by investigation of its solvable extensions.

We find that the extension of a solvable algebra with the nilradical $\tilde{n}_{n-2,1}$ to a solvable extension of the nilradical $n_{n,3}$ is unique when $d_n \neq 0$ and $c_{n-1} \neq 0$; otherwise, several non–equivalent extensions do exist.

The case when none of the conjugate derivations $D_\Phi$ leaves $\tilde{n}_{n-2,1}$ invariant necessarily means $d_n = c_{n-1} \rightarrow 1$, $d_{n-1} \neq 0$ and leads to a unique solvable algebra $s_{n+1,9}$ in the list below.
Any solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathfrak{n}_{n,3}$ has the dimension $\dim \mathfrak{s} = n + 1$, or $\dim \mathfrak{s} = n + 2$. I.e. the bound (14) is not saturated here.

Five types of solvable Lie algebras of dimension $\dim \mathfrak{s} = n + 1$ with the nilradical $\mathfrak{n}_{n,3}$ exist for any $n \geq 7$. They are represented by the following:
\[ [f, e_1] = (\alpha + 2\beta)e_1, \]
\[ [f, e_2] = 2\beta e_2, \]
\[ [f, e_3] = (\alpha + \beta)e_3, \]
\[ [f, e_k] = ((3 - k)\alpha + 2\beta)e_k, \quad 4 \leq k \leq n - 2, \]
\[ [f, e_{n-1}] = \beta e_{n-1}, \]
\[ [f, e_n] = \alpha e_n. \]
The classes of mutually nonisomorphic algebras of this type are

\[ s_{n+1,1}(\beta) : \quad \alpha = 1, \beta \in F \setminus \{0, -\frac{1}{2}, \frac{n-5}{2}\}, \]

\[ s_{n+1,2} : \quad \alpha = 1, \beta = \frac{n-5}{2}, \]

\[ s_{n+1,3} : \quad \alpha = 1, \beta = 0, \]

\[ s_{n+1,4} : \quad \alpha = 1, \beta = -\frac{1}{2}, \]

\[ s_{n+1,5} : \quad \alpha = 0, \beta = 1, \]

where the splitting into subcases reflects different dimensions of the characteristic series.
\( s_{n+1,6}(\epsilon) : \]

\[
\begin{align*}
[f, e_1] &= (n - 3)e_1, \\
[f, e_2] &= (n - 4)e_2, \\
[f, e_3] &= (1 + \frac{n - 4}{2})e_3, \\
[f, e_k] &= (n - 1 - k)e_k, \quad 4 \leq k \leq n - 2, \\
[f, e_{n-1}] &= \frac{n - 4}{2}e_{n-1}, \\
[f, e_n] &= e_n + \epsilon e_{n-2}
\end{align*}
\]

where \( \epsilon = 1 \) over \( \mathbb{C} \), whereas over \( \mathbb{R} \) \( \epsilon = 1 \) for \( n \) odd, \( \epsilon = \pm 1 \) for \( n \) even.
$s_{n+1,7} : \quad [f, e_1] = e_1,
[f, e_2] = 0,
[f, e_3] = e_3 - e_1,
[f, e_k] = (3 - k)e_k, \quad 4 \leq k \leq n - 2,
[f, e_{n-1}] = e_2,
[f, e_n] = e_n.$
\[ s_{n+1,8}(a_2, a_3, \ldots, a_{n-3}) : \]
\[
[f, e_1] = e_1, \quad [f, e_2] = e_2, \]
\[
[f, e_3] = \frac{1}{2}e_3, \]
\[
[f, e_k] = e_k + \sum_{l=4}^{k-2} a_{k-l+1}e_l + a_{k-2}e_2 + a_{k-1}e_1, \quad 4 \leq k \leq n - 2, \]
\[
[f, e_{n-1}] = \frac{1}{2}e_{n-1} + a_2e_3, \]
\[
[f, e_n] = 0, \]

\( a_j \in \mathbb{F} \), at least one \( a_j \) satisfies \( a_j \neq 0 \). Over \( \mathbb{C} \): the first \( a_j \neq 0 \) satisfies \( a_j = 1 \). Over \( \mathbb{R} \): the first \( a_j \neq 0 \) for even \( j \) satisfies \( a_j = 1 \). If all \( a_j = 0 \) for \( j \) even, then the first \( a_j \neq 0 \) (\( j \) odd) satisfies \( a_j = \pm 1 \).
\( s_{n+1,9} : \)

\[ [f, e_1] = 3e_1, \]

\[ [f, e_2] = 2e_2, \]

\[ [f, e_3] = 2e_3 - e_2, \]

\[ [f, e_k] = (5 - k)e_k, \quad 4 \leq k \leq n - 2, \]

\[ [f, e_{n-1}] = e_{n-1} + e_4, \]

\[ [f, e_n] = e_n + e_{n-1}. \]
Precisely one solvable Lie algebra $\mathfrak{s}_{n+2}$ of $\dim \mathfrak{s} = n + 2$ with the nilradical $\mathfrak{n}_{n,3}$ exists. It is presented in a basis $(e_1, \ldots, e_n, f_1, f_2)$ where the Lie brackets involving $f_1$ and $f_2$ are

\[
\begin{align*}
[f_1, e_1] &= e_1, & [f_2, e_1] &= 2e_1, \\
[f_1, e_2] &= 0, & [f_2, e_2] &= 2e_2, \\
[f_1, e_k] &= (3 - k)e_k, & [f_2, e_k] &= 2e_k, & 4 \leq k \leq n - 2, \\
[f_1, e_{n-1}] &= 0, & [f_2, e_{n-1}] &= e_{n-1}, \\
[f_1, e_n] &= e_n, & [f_2, e_n] &= 0, & [f_1, f_2] = 0.
\end{align*}
\]

For $n = 5, 6$ the results are slightly different.
Conclusions

- We have reviewed our current knowledge concerning the classifications of solvable Lie algebras.
- We have introduced an improved upper estimate on the number of nonnilpotent generators of such algebras.
- We have presented the results of one particular classification of solvable extensions of $\mathfrak{n}_{n,3}$ which was constructed by means of already known classification for a different nilradical, namely $\mathfrak{n}_{n-2,1}$. 
Thank you for your attention