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Prof. Peter Webb
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Name of faculty adviser

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Signature of faculty adviser

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Date

GRADUATE SCHOOL

Homological Properties of Category Algebras

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Contents

Chapter 1

Introduction

In this thesis, we introduce and study the notion of a category algebra, denoted by $R\mathcal{C}$, where R is the base ring and $\mathcal{C}$ is some small category. This is a type of associative algebra, which simultaneously generalizes several important constructions in representation theory and combinatorics, notably the path algebra of a quiver, the incidence algebra of a poset, and the group algebra of a group. Representations of small categories arise when we consider any diagram of modules, or take inverse or direct limits. We are greatly motivated by the representations of certain categories constructed from subgroups of a group, which have been intensively studied in recent years and are currently the subject of active investigation, see Broto-Levi-Oliver [6], Dwyer [10], Grodal [17], Jackowski-McClure-Oliver [24], Linckelmann [27] and Symonds [37]. As an example we mention the work in which classifying spaces of groups are approximated by homotopy colimits of certain diagrams of spaces, indexed by categories made of certain subgroups of the groups. The homology of such a homotopy colimit may be calculated using the Bousfield-Kan spectral sequence, whose E_2 page is expressed in terms of higher limits. We mention also as a motivation the constructions of Ronan and Smith (described in Benson [5]), in which they construct irreducible modules for finite groups from diagrams of other modules.

Throughout this thesis, we assume R is a commutative ring with an identity. Let $\mathcal{C}$ be a small category. A representation of $\mathcal{C}$ over R is defined to be a (covariant) functor from $\mathcal{C}$ to R-mod. In other words, a representation of $\mathcal{C}$ over R is an object of the functor category $R\text{-mod}^{\mathcal{C}}$, which is well-known to be abelian and has enough projectives and injectives. There is a result, due to Mitchell [28], saying that if $\mathcal{C}$ has finitely many objects one can identify the category of representations of $\mathcal{C}$ with the category of (left) $R\mathcal{C}$-modules (Proposition 2.3.2). Because of this, any representation $M : \mathcal{C} \to R$-mod will be called an $R\mathcal{C}$-module. There is a prominent module, the trivial module or constant functor $\underline{R} \in R\mathcal{C}$-mod, which plays an important role in our research, sending every object of $\mathcal{C}$ to R and every morphism to the identity. When we study the representations of a category $\mathcal{C}$, it is natural to think of the homomorphisms $\mathrm{Hom}_{R\mathcal{C}}(M, N)$ between two modules M and N. The computation of the values $\mathrm{Ext}^*_{R\mathcal{C}}(M, N)$ of the derived functors of $\mathrm{Hom}_{R\mathcal{C}}(M, -)$ at N (or equivalently the values of the derived functors of $\mathrm{Hom}_{R\mathcal{C}}(-, N)$ at M) will be a focus of this thesis. When $M = \underline{R}$, the groups $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N)$ can be identified with $\varprojlim^*_{\mathcal{C}} N$, the higher inverse limits of N over $\mathcal{C}$. The identification reveals the connections between the results in this thesis and those in the papers mentioned in the first paragraph, where computing higher limits is one of the key ingredients. The higher limits of a functor N over a category $\mathcal{C}$ can be thought as the Baues-Wirsching cohomology groups $\mathrm{H}^*(\mathcal{C}, N)$ of $\mathcal{C}$, because of the well-known isomorphism $\mathrm{H}^*(\mathcal{C}, N) \cong \varprojlim^*_{\mathcal{C}} N$. It's worthy of noting that when $N = \underline{R}$, we have $\varprojlim^*_{\mathcal{C}} \underline{R} \cong \mathrm{H}^*(\mathcal{C}, \underline{R}) \cong \mathrm{H}^*(|\mathcal{C}|, R)$, where $|\mathcal{C}|$ is the classifying space of $\mathcal{C}$.

The usual way of computing $\mathrm{Ext}^*_{R\mathcal{C}}(M, N)$ is to take a projective resolution of M, i.e. $\mathcal{P} \to M \to 0$, and then the homology of the cochain complex $0 \to \mathrm{Hom}_{R\mathcal{C}}(\mathcal{P}, N)$ gives rise to the desired Ext groups. Similarly one can also use an injective resolution of N to obtain a cochain complex and to compute these Ext groups, but this is less used in practice under our circumstances. In general, the explicit computation of the Ext groups is very hard, and thus many researchers are looking for principles of reducing the computation of the Ext groups over $\mathcal{C}$ to that over some simpler category $\mathcal{D}$, which is often a (full) subcategory of $\mathcal{C}$. In light of this, searching for reduction formulas is also our direction on computing Ext groups in this thesis. We comment here that when $\mathcal{C}$ is a category associated to a finite group (e.g. an orbit category), some nice reduction formulas have been achieved by various authors, see for example the work of Grodal [17]. In that case, group action is heavily used throughout the calculations, and this is not the approach of us because we consider general small categories. In the general context, there is a result of Jackowski and Słomińska [25] (Corollary 3.2.7) which establishes an isomorphism between $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M)$ and $\mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, \mathrm{Res}_\mu M)$ for a functor $\mu : \mathcal{D} \to \mathcal{C}$ satisfying the conditions that the overcategories associated to μ (Definition 3.2.2) are all R-acyclic (that is, the reduced homology groups of the classifying spaces of the overcategories with coeffecients in R vanish).

We begin with the observation that Jackowski and Słomińska's result can be established within a purely representation-theoretic setting, because their proof uses the left Kan extension of $\mu : \mathcal{D} \to \mathcal{C}$, which can be identified with the induction $\uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} -$ (when $R\mathcal{C}$ is a right $R\mathcal{D}$-module) since they both are the left adjoints of the restriction $\mathrm{Res}_\mu : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$. From this new angle, the R-acyclicity of the overcategories associated to $\mu : \mathcal{D} \to \mathcal{C}$ can be obtained by requiring the functor $R\mathcal{C} \otimes_{R\mathcal{D}} -$ to be exact (Lemma 3.3.3) (i.e. the right $R\mathcal{D}$-module $R\mathcal{C}$ is flat) and having an isomorphism $R\mathcal{C} \otimes_{R\mathcal{D}} \underline{R} \cong \underline{R}$. As an effort to get the flatness of $R\mathcal{C}$, one can choose suitable $\mathcal{D}$ and $\mu : \mathcal{D} \to \mathcal{C}$ so that $R\mathcal{C}$ becomes a right projective $R\mathcal{D}$-module. Sorting out the projectivity of $R\mathcal{C}$ as an $R\mathcal{D}$-module is easier than examining the R-acyclicity of the overcategories associated to $\mu : \mathcal{D} \to \mathcal{C}$, especially when $\mathcal{D}$ is a (full) subcategory of $\mathcal{C}$. There is a lot of information about $R\mathcal{C}$-mod in the book of tom Dieck [9], where many results are attributed to Lück [26]. When $\mathcal{C}$ is a finite EI-category, Lück has classified the simple and projective modules of $R\mathcal{C}$, while tom Dieck has listed equivalent conditions for $R\mathcal{C}$ being projective as an $R\mathcal{D}$-module.

One who knows the Eckmann-Shapiro Lemma (see Benson [4]) in the cohomology theory of algebras would compare the isomorphism of Jackowski and Słomińska with the well-known isomorphism $\mathrm{Ext}^*_{R\mathcal{C}}(M \uparrow_{\mathcal{D}}^{\mathcal{C}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M, N \downarrow_{\mathcal{D}}^{\mathcal{C}})$, assuming that $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module (in representation theory, the restriction is normally written as $\downarrow_{\mathcal{D}}^{\mathcal{C}}$ instead of Res_μ). Intuitively, by replacing M by $\underline{R}$ in the Eckmann-Shapiro Lemma one gets immediately something similar to the isomorphism of Jackowski and Słomińska. Furthermore if the $R\mathcal{C}$-module $\underline{R}$ can be expressed as $\underline{R} \uparrow_{\mathcal{D}}^{\mathcal{C}}$, an induced module from the $R\mathcal{D}$-module $\underline{R}$, then one has exactly the the isomorphism of Jackowski and Słomińska. As a matter of fact, we show that every indecomposable $R\mathcal{C}$-module is an induced module, and as a special case we always have $\underline{R} \cong \underline{R} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ for some subcategory $\mathcal{D}$. This is a result from our theory of vertices and sources for category algebras, which tells us a way to parameterize the indecomposable $R\mathcal{C}$-modules according to their vertices. From the same theory one can show that every group

$\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$ is isomorphic to $\mathrm{Ext}^*_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$ for some subcategory $\mathcal{D}$, if $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module.

The layout of this thesis is described as follows. In Chapter 2, we define category algebras and give some examples. In Chapter 3 we introduce the cohomology theory of small categories $H^*(\mathcal{C};M)$, where $\mathcal{C}$ is a small category and M is an $R\mathcal{C}$-module. The definition is followed by a proof of the fundamental isomorphisms $\varprojlim^*_{\mathcal{C}} M \cong \mathrm{H}^*(\mathcal{C};M) \cong \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},M)$. Then we collect some interesting properties of the cohomology and representations of small categories, as well as existing isomorphisms that reduce the computation of the groups $\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$. We spend Section 3.3 to explore the isomorphism of Jackowski and Słomińska, and then give it an algebraic interpretation as we mentioned above.

From Chapter 4, we focus on the so-called EI-categories, in which every endomorphism is an isomorphism. The most important results in this thesis are obtained when we work with EI-categories. We need to point out that many categories occurred in the representation and cohomology theory of finite groups fall into this class, and we may readily list posets, groups, the Frobenius categories and the orbit categories associated to finite groups as major examples. At the beginning of Chapter 4, we provide a complete description of the projective and simple $R\mathcal{C}$-modules, which was achieved by Lück [26] (also see tom Dieck [9]). The simple $R\mathcal{C}$-modules are usually denoted by $S_{x,V}$ because each of them is determined by a simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module V for some object $x \in \mathrm{Ob}\,\mathcal{C}$. Consequently, the projective cover of such a simple is written as $P_{x,V}$. Note that in order to invoke the Krull-Schmidt Theorem, we normally assume the base ring R is a field or a complete local ring. With the description of indecomposable projectives, we can show when $\underline{R}$ is projective (Proposition 4.1.5). The next step is to tell when the restriction $\downarrow^{\mathcal{C}}_{\mathcal{D}} : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ preserves (left or right) projectives, where $\mathcal{D}$ is a subcategory of $\mathcal{C}$. The main results were obtained by tom Dieck [9], though we wrote out these results before knowing his. The core of Chapter 4 is the development of our theory of vertices and sources, which is divided into two sections. The first one gives some general results about the theory of relative projectivity for category algebras, and the second one deals with full subcategories only. Let M be an $R\mathcal{C}$-module and $\mathcal{D} \subset \mathcal{C}$ a subcategory. Then M is $\mathcal{D}$-projective if M is isomorphic to a direct summand of $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$. A key fact is that if M is projective relative to a full subcategory $\mathcal{D}$, then $M\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is indecomposable and $M \cong M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$. This enables us to prove that the category of $R\mathcal{C}$-modules which are projective relative to a full subcategory $\mathcal{D}$ is equivalent to the category of $R\mathcal{D}$-modules. We also prove that for an indecomposable $R\mathcal{C}$-module M it's possible to find the smallest convex subcategory relative to which M is projective. This subcategory, $\mathcal{V}_M$, will be called the vertex of M. Centered at this fact, we develop our theory of vertices and sources, which functions in a similar way as its counterpart in group representation theory. At the end of Chapter 4, we consider two things that are of particular interest to us. Firstly we want to know more about the vertex of the trivial module $\underline{R}$ because we always have higher limits in mind. Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ such that $\underline{R}$ is $\mathcal{D}$-projective. We obtain a characterization of $\mathcal{D}$ through the overcategories associated with the inclusion. As an alternative description of $\mathcal{D}$, we also show $\mathcal{D}$ must contain the so-called weakly essential objects of $\mathcal{C}$ (Corollary 4.5.5). Secondly, we realize that when $\mathcal{C}$ is a poset the subcategory $\mathcal{V}_{\underline{R}}$ is totally determined by the weakly essential objects. Since every EI-category has an underlying poset, we want to know whether this statement remains true for other EI-categories. Since the proof for the statement for posets uses a theorem of Bouc (Proposition 4.6.6), we generalize his theorem to certain types of EI-categories before we can make further observations. The categories in which we are interested are the categories with subobjects, defined in Section 4.6. Any category with subobjects is written as $(\mathcal{C}, \mathcal{I})$, where $\mathcal{I}$ is a special subcategory of $\mathcal{C}$ controls many homological properties of $\mathcal{C}$. We prove any finite category with subobjects $(\mathcal{C}, \mathcal{I})$ is an EI-category so that it makes sense to study $\mathcal{V}_{\underline{R}}$. We show when $(\mathcal{C}, \mathcal{I})$ is a finite category with subobjects (e.g. posets), then $\mathcal{I}$ is a poset and the full subcategory of $\mathcal{C}$ consisting of weakly essential objects of $\mathcal{I}$ is the smallest full subcategory, relative to which $\underline{R}$ is projective (Corollary 4.6.9). Thus the vertex of $\underline{R}$ is the "convex hull" of the full subcategory made up of all weakly essential objects of $\mathcal{I}$.

In Chapter 5 we investigate some particular projective resolutions for $R\mathcal{C}$-modules, which leads us to certain aspects of the representation and cohomology theory of categories, as well as some supplementary methods for computing Ext groups other than those shown in Chapter 4. These methods work very well when dealing with $\mathrm{Ext}^*_{R\mathcal{C}}(M, N)$ for two atomic modules. The most significant example is the calculations of $\mathrm{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W})$, where $\mathcal{C}$ can be replaced by the full subcategory consisting of objects "between" x and y. When V and W are equal to R, these Ext groups can be identified with cohomology groups of some topological object. Also in Chapter 5, we use minimal resolutions of the simples to show when a category algebra has finite global dimension. As a vehicle to study higher limits, we investigate the cohomology ring, $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R})$, of an EI-category $\mathcal{C}$ in Chapter 6. We find that the cohomology ring $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R})$ is not finitely generated in general. Instead we manage to prove that the cohomology ring modulo a nilpotent ideal is isomorphic to a subring of the cohomology ring of a finite group. We conjecture that this subring is finitely generated, and this is still an open question. Finally, we put an example in Chapter 7 to illustrate our main results obtained in this thesis.

This thesis is typeset by LaTeX 2_ε, while all the commutative diagrams are drawn by using the package XY-pic.

Chapter 2

Category Algebras: Definitions and Examples

Throughout this thesis, the base ring R is always a commutative ring with an identity. A module will be a finitely generated left module, if it is not otherwise specified.

2.1 Definitions

Definition 2.1.1. *Let $\mathcal{C}$ be a category and R a commutative ring. The category algebra $R\mathcal{C}$ is the free R-module whose basis is the set of morphisms of $\mathcal{C}$. We define a product on the basis elements of $R\mathcal{C}$ by*

$$f * g = \begin{cases} f \circ g, & \text{if } f \text{ and } g \text{ can be composed in } \mathcal{C} \\ 0\ , & \text{otherwise} \end{cases}$$

and then extend this product linearly to all elements of $R\mathcal{C}$. With this product, $R\mathcal{C}$ becomes an associative R-algebra.

A category $\mathcal{C}$ is said to be finite if its morphisms form a finite set, denoted by $\mathrm{Mor}(\mathcal{C})$. Note that this implies the set of objects $\mathrm{Ob}\,\mathcal{C}$ is finite. Also note that $R\mathcal{C}$ is of finite R-rank if and only if $\mathcal{C}$ is finite.

Remark 2.1.2. *If $\mathrm{Mor}(\mathcal{C})$ is finite, then $\sum_{x\in\mathrm{Ob}\,\mathcal{C}} 1_x$ is the identity of $R\mathcal{C}$, where 1_x is the identity of $\mathrm{Aut}_{\mathcal{C}}(x)$ for every $x \in \mathrm{Ob}\,\mathcal{C}$.*

We say a category $\mathcal{C}$ is connected if $\mathcal{C}$ as a (directed) graph is connected. Every category $\mathcal{C}$ is a disjoint union of connected components $\mathcal{C} = \cup_{i \in J} \mathcal{C}_i$, where each $\mathcal{C}_i$ is connected and J is an index set. Consequently the category algebra $R\mathcal{C}$ becomes a direct sum of ideals $R\mathcal{C}_i, i \in J$, and thus in order to study the properties of $R\mathcal{C}$ it suffices to study the properties of each $R\mathcal{C}_i$. For simplicity we make the following assumption.

Convention *In this thesis, we assume $\mathcal{C}$ is connected.*

We shall show that a fundamental property of a category algebra $R\mathcal{C}$ is that it provides a mechanism for discussing representations of $\mathcal{C}$, in a sense which we now define.

Definition 2.1.3. *Let $\mathcal{C}$ be a small category. A representation of $\mathcal{C}$ over a commutative ring R is a (covariant) functor $M : \mathcal{C} \to R$-mod.*

The following result was discovered by Mitchell [28].

Proposition 2.1.4. *For any category $\mathcal{C}$ with finitely many objects, its representations can be identified with unital $R\mathcal{C}$-modules.*

Proof. Assume $M : \mathcal{C} \to R$-mod is a representation of $\mathcal{C}$. We construct an R-module $\tilde{M} = \oplus_{x \in \mathrm{Ob}\,\mathcal{C}} M(x)$, and show it has a natural $R\mathcal{C}$-module structure. For any $m \in M(x)$ and any morphism $f \in \mathrm{Mor}(\mathcal{C})$, we define $f \cdot m = M(f)(m)$, if $f \in \mathrm{Hom}_{\mathcal{C}}(x, x')$ for some $x' \in \mathrm{Ob}\,\mathcal{C}$. Otherwise, we define $f \cdot m = 0$. By extending this operation linearly to all $m \in \tilde{M}$ and $f \in R\mathcal{C}$ we establish an $R\mathcal{C}$-module structure on $\tilde{M}$. On the other hand, if $\tilde{M}$ is an $R\mathcal{C}$-module, we may construct a representation M of $\mathcal{C}$ by defining $M(x) = 1_x \cdot M$. We observe that if $f \in \mathrm{Hom}_{\mathcal{C}}(x, x')$ and $m \in 1_x \cdot M$, then $f \cdot m = (1_{x'} \circ f) \cdot m = 1_{x'} \cdot (f \cdot m) \in 1_{x'} \cdot M$ and from the axioms for $\tilde{M}$ to be an $R\mathcal{C}$-module we see that $M : \mathcal{C} \to R$-mod is a functor. We may check these two constructions are inverse to each other, up to natural isomorphism. It follows that the category of representations of $\mathcal{C}$ and that of $R\mathcal{C}$-modules, are equivalent. □

Because of the above proposition, we will mostly talk about $R\mathcal{C}$-modules throughout this thesis, which is very common in representation theory. There is a distinguished module that plays an important role in the representation and cohomology theory of small categories. The trivial module (or the constant functor) $\underline{R}$ is defined by $\underline{R}(x) = R$ for all $x \in \operatorname{Ob}\mathcal{C}$ and $\underline{R}(f) = \operatorname{Id}$ for all $f \in \operatorname{Mor}\mathcal{C}$. We will see shortly that it is a generalization of the trivial module for group algebras to category algebras.

2.2 Examples

We provide three examples which give us the inspiration of defining category algebras. Our first example asserts that quiver algebras are category algebras.

Definition 2.2.1. *A quiver* $\mathfrak{q} = (\Gamma_0, \Gamma_1)$ *is a directed graph having a set of vertices* Γ_0 *and a set of arrows* Γ_1.

The path algebra of a quiver may be defined in terms of the free category generated by the quiver.

Definition 2.2.2. *(Mac Lane [29]) Any directed graph* $\mathfrak{g} = (\Gamma_0, \Gamma_1)$ *may be used to generate a category* $\mathcal{C}_{\mathfrak{g}}$ *on the same set* Γ_0 *of objects; the morphisms of this category will be the strings of composable arrows of* $\mathfrak{g}$. *It is called the free category generated by* $\mathfrak{g}$.

Example 2.2.3. *For the directed graph* $\mathfrak{g} : x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ *with objects* $\{x, y, z\}$ *and arrows* $\{\alpha, \beta\}$, $\mathcal{C}_{\mathfrak{g}}$ *is the category with the same objects* $\{x, y, z\}$ *and with the six morphisms* $\{1_x, 1_y, 1_z, \alpha, \beta, \beta\alpha\}$. *The category algebra* $R\mathcal{C}_{\mathfrak{g}}$ *has R-rank 6. In fact,* $R\mathcal{C}_{\mathfrak{g}}$ *is isomorphic to the algebra of* 3×3 *upper triangular matrices, because we can pose the following assignments:* $1_x \mapsto E_{33}, 1_y \mapsto E_{22}, 1_z \mapsto E_{11}, \alpha \mapsto E_{23}, \beta \mapsto E_{12}$ *and* $\beta\alpha \mapsto E_{13}$, *where* E_{ij} *is an elementary* 3×3*-matrix. Since these elementary matrices multiply in the same way as the basis elements of* $R\mathcal{C}_{\mathfrak{g}}$, *we see that these assignments extend linearly to an algebra homomorphism.*

Definition 2.2.4. *The path algebra of a quiver* $\mathfrak{q}$ *is the category algebra* $R\mathcal{C}_{\mathfrak{q}}$ *of the free category* $\mathcal{C}_{\mathfrak{q}}$.

The above definition is equivalent to the usual definition of the path algebra of a quiver. There is an extensive literature on the role of quivers in representation theory, and we mention only Auslander-Reiten-Smalø ([2] III.1) for background. The following result is a direct consequence of Definition 2.2.4.

Lemma 2.2.5. *The modules of the path algebra of a finite quiver can be identified with the functors from the free category generated by this quiver to R-mod.*

It is necessary to point out that, given a category, its category algebra is usually different from its path algebra. Nevertheless, there is a relationship between these two algebras for a category.

Proposition 2.2.6. *Let* q *be a category. We may regard* q *as a quiver and form the free category* $\mathcal{C}_{\mathsf{q}}$ *over* q*. There is a natural functor* $\phi : \mathcal{C}_{\mathsf{q}} \to \mathsf{q}$*, which extends to a surjective homomorphism, still denoted by* $\phi : R\mathcal{C}_{\mathsf{q}} \to R\mathsf{q}$*, from the path algebra of* q *to the category algebra of* q*, such that its kernel* I *is generated by* $\{\overset{\alpha_1}{\leftarrow}\overset{\alpha_2}{\leftarrow} - \overset{\alpha_1\alpha_2}{\longleftarrow}\}$*, where* α_1 *and* α_2 *are arrows of* q*. This epimorphism induces a natural isomorphism of R-algebras* $R\mathcal{C}_{\mathsf{q}}/I$ *and* $R\mathsf{q}$*.*

Proof. The functor ϕ is defined as follows. For each $x \in \mathrm{Ob}\,\mathcal{C}_{\mathsf{q}}$, we let $\phi(x) = x$. For each $\alpha \in \mathrm{Mor}\,\mathcal{C}_{\mathsf{q}}$, we let $\phi(\alpha) =$ the composite of the maps in the string α. The functor can be extended linearly to an epimorphism $\Phi : R\mathcal{C}_{\mathsf{q}} \to R\mathsf{q}$ with a kernel equal to I so the statement follows. □

Our second example says the incidence algebra of a poset (partially ordered set) is a category algebra.

Definition 2.2.7. *Let* Γ *be a finite poset and* R *a commutative ring. The incidence algebra of* Γ*, consists of all incidence functions*

$$A(\Gamma) = \{f \mid \Gamma \times \Gamma \to R \text{ with } f(x,y) = 0 \text{ if } x \not\leq y\}$$

by taking point-wise summation and scalar multiplication of such functions as sums and scalar product

$$(f+g)(x,y) = f(x,y) + g(x,y)$$

$$(\alpha f)(x,y) = \alpha f(x,y), \alpha \in R$$

and by defining the product $h = f * g$ *as*

$$(f * g)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$$

Note that the Kronecker delta function $\delta(x,y)$ *is the two-sided identity of* $A(\Gamma)$*.*

Let Γ be a poset. We can define a category, still named Γ, whose objects are the vertices of the poset. For any two objects x and y in Γ, there exists a (unique) morphism in $\mathrm{Hom}_\Gamma(x, y)$ if and only if $x \leq y$. We show the incidence algebra $A(\Gamma)$ of Γ is naturally isomorphic to its category algebra $R\Gamma$ by defining an incidence function for every morphism in Γ as follows. If $\alpha \in \mathrm{Hom}_\Gamma(x, y)$ is the unique morphism with domain x and co-domain y, then we define an incidence function $f_\alpha(*, *) \in A(\Gamma)$ by requiring $f_\alpha(x, y) = 1$ and $f_\alpha(*, *) = 0$ otherwise.

Proposition 2.2.8. *The incidence functions f_α, $\alpha \in \mathrm{Mor}(\Gamma)$, form a basis of the incidence algebra $A(\Gamma)$ of Γ. There exists a natural isomorphism between the incidence algebra $A(\Gamma)$ and the category algebra $R\Gamma$.*

Proof. Let $f \in A(\Gamma)$. We can define a map sending f to $\sum_{\alpha:x \to y} f(x, y)\alpha$. This is an isomorphism. □

The last example shows that the notion of a category algebra also generalizes the notion of a group algebra.

Example 2.2.9. *Let G be a (discrete) group and R a commutative ring. The group algebra RG is additively a free R-module with a basis the elements of G. Let $\hat{G}$ be the category with only one object $*$ whose morphisms are the elements of G. We can see the group algebra of G is a category algebra, i.e. $RG = R\hat{G}$. By definition a left RG-module M is an R-module equipped with a group homomorphism $\Phi : G \to \mathrm{Aut}_R(M)$. This can be interpreted as a representation of the category $\hat{G}$, which sends $*$ to $M \in R$-mod and $\mathrm{Aut}_{\hat{G}}(*) = G$ into $\mathrm{Aut}_R(M)$.*

It's easy to see that $\underline{R}$ is exactly the trivial module of the group algebra $RG = R\hat{G}$.

Chapter 3

Basic Homological Properties

In this chapter we introduce the cohomology theory of small categories which can be used to define and calculate the higher (inverse) limits of an $R\mathcal{C}$-module M over $\mathcal{C}$. After giving the definition, we will establish the connection between the cohomology theory of small categories with that of category algebras. Then we will provide formulas for computing these cohomology groups.

3.1 Cohomology theory of small categories

The cohomology theory of small categories has been discussed in various places in the literature, and readers are referred to Baues-Wirsching [3], Generalov [15] and Oliver [31] for more details. One can also find in Gabriel-Zisman [14] and Hilton-Stammbach [21] the homology theory of small categories.

Definition 3.1.1. *Let $\mathcal{C}$ be a small category and R a commutative ring. We define the n-th cohomology group $\mathrm{H}^n(\mathcal{C}, M)$ of $\mathcal{C}$ with coefficients in a covariant functor $M : \mathcal{C} \to R$-mod to be the n-th homology group of the following cochain complex $\{C^*(\mathcal{C}, M), \delta\}$, where*

$$C^n(\mathcal{C}, M) = \{f \mid f : N_n(\mathcal{C}) \to \prod_{[x_0 \to x_1 \to \cdots \to x_n]} M(x_n), \text{ and } f([x_0 \to \cdots \to x_n]) \in M(x_n)\}$$

for all $n \geq 0$, with $N(\mathcal{C}) = \{N_n(\mathcal{C})\}$ the nerve of $\mathcal{C}$, and where for any $f \in C^n(\mathcal{C}, M)$,

$$\delta(f)([x_0 \to \cdots \to x_n \xrightarrow{\phi} x_{n+1}]) = \sum_{i=0}^{n} (-1)^i f([x_0 \to \cdots \to \hat{x_i} \to \cdots \to x_{n+1}])$$

$$+(-1)^{n+1}M(\phi)(f([x_0 \to \cdots \to x_n])).$$

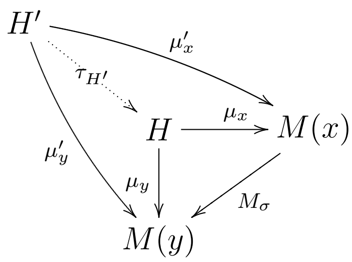
The next result was originally proved by Gabriel-Zisman [14] and J.-E. Roos [36]. Our proof for it is taken from Oliver [31].

Proposition 3.1.2. *Let $\mathcal{C}$ be a small category and R a commutative ring. If $M : \mathcal{C} \to R$-mod is a functor, then*

$$\varprojlim^i_{\mathcal{C}} M \cong \mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, M) \cong \mathrm{H}^i(\mathcal{C}, M),$$

for any $i \geq 0$.

Proof. The first identities can be proved by using the definition of the inverse limit of a functor M (see Mac Lane [29]). Let $H = \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$ and ΔH be the diagonal functor from $\mathcal{C}$ to R-mod such that $\Delta H(x) = H$ for all $x \in \mathrm{Ob}\,\mathcal{C}$. In order to show H is the inverse limit of M, we have to construct a natural transformation $\mu : \Delta H \to M$ and show it's universal. Firstly, we define μ_x for any $x \in \mathrm{Ob}\,\mathcal{C}$ by $\mu_x(f) = f_x(1)$ for all $f \in \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M) = H = \Delta H(x)$. It's easy to check that for any $\sigma \in \mathrm{Hom}_{\mathcal{C}}(x, y), \mu_y = M_\sigma \circ \mu_x$. Hence μ is a natural transformation. Secondly, for any other H' equipped with a natural transformation $\mu' : \Delta H' \to M$, we need to show there exits a unique group homomorphism $\tau_{H'} : H' \to H$ which is compatible with the natural transformations μ and μ'. For each $h' \in H'$, we define $\tau_{H'}(h') = g^{h'}$ where $g^{h'} \in H = \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$ is defined by $g^{h'}(1_x) = {\mu'}_x(h')$ for all $x \in \mathrm{Ob}\,\mathcal{C}$. We can verify that $\tau_{H'}$ is well-defined and that it is the unique homomorphism making the following diagram commutative.



Thus we proved $\varprojlim_{\mathcal{C}} M \cong \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$. Since the kernel of $\delta : C^0(\mathcal{C}, M) \to C^1(\mathcal{C}, M)$ can be identified with $\mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$, we get $\mathrm{H}^0(\mathcal{C}, M) \cong \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$.

Now let's prove the isomorphisms for higher groups. Since $\varprojlim_{\mathcal{C}} - \cong \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, -)$ is left exact and $\varprojlim^i_{\mathcal{C}} -$ is its i-th right derived functor, we have $\varprojlim^i_{\mathcal{C}} M \cong \mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, M)$

and so it remains to show $\mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, M) \cong \mathrm{H}^i(\mathcal{C}, M)$. Because $\mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, M)$ can be computed by using any projective resolution of $\underline{R}$, we can construct a particular projective resolution named $(\mathcal{P} = \{P_n\}_{n=-1}^{\infty}, \sigma)$ to calculate it. For each $n \geq -1$, we define a functor $P_n : \mathcal{C} \to R$-mod as follows. For any $x \in \mathrm{Ob}\,\mathcal{C}$, let $P_n(x)$ be the free abelian group with a basis the set of all sequences $[x_0 \to x_1 \to \cdots \to x_n \to x]$ of morphisms in $\mathrm{Mor}(\mathcal{C})$ ending in x. For any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$, $P_n(f)$ is defined by its action on base elements: $[x_0 \to \cdots \to x_n \xrightarrow{\phi} x] \mapsto [x_0 \to \cdots \to x_n \xrightarrow{f\phi} y]$. Next we define the boundary map $\sigma = \{\sigma_x\} : P_n \to P_{n-1}$ by setting

$$\sigma_x([x_0 \to x_1 \to \cdots \to x_n \to x]) = \sum_{i=0}^{n} (-1)^i [x_0 \to \cdots \to \hat{x}_i \to \cdots \to x_n \to x],$$

on base elements. It's easy to check that, for each x, the chain complex

$$\cdots \to P_2(x) \to P_1(x) \to P_0(x) \to P_{-1}(x) \cong \underline{R} \to 0$$

is split by the maps $[\cdots \to x_n \to x] \to [\cdots \to x_n \to x \xrightarrow{\mathrm{Id}} x]$, and hence is exact. In order to complete our proof, we have to show that $(\mathcal{P}, \sigma)$ is a projective resolution of $\underline{R}$, and that $\{\mathrm{Hom}_{R\mathcal{C}}(P_n, M), \sigma^*\}$ is isomorphic to the complex $\{C^n(\mathcal{C}, M), \delta\}$ for any functor M. In fact we shall establish the following isomorphisms of complexes

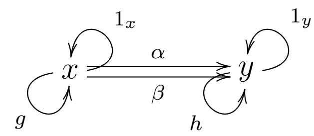
$$\{\mathrm{Hom}_{R\mathcal{C}}(P_n, M)\} \cong \{C^n(\mathcal{C}, M)\} \cong \{\prod_{x_0 \to \cdots \to x_n} M(x_n)\},$$

and then we can see $\mathrm{Hom}_{R\mathcal{C}}(P_n, -)$ is exact and hence P_n is a projective $R\mathcal{C}$-module. Since the second isomorphism is natural, we only prove the first here. For any natural transformation $\theta \in \mathrm{Hom}_{R\mathcal{C}}(P_n, M)$, we define a $f \in C^n(\mathcal{C}, M)$ by $f([x_0 \to \cdots \to x_n]) = \theta_{x_n}([x_0 \to \cdots \to x_n \xrightarrow{\mathrm{Id}} x_n])$. On the other hand, for any $f \in C^n(\mathcal{C}, M)$, we can construct a $\theta \in \mathrm{Hom}_{R\mathcal{C}}(P_n, M)$, such that $\theta_x([x_0 \to \cdots \to x_n \xrightarrow{\phi} x]) = M(\phi) f([x_0 \to \cdots \to x_n])$. One can easily check that these two constructions are mutually inverse, and commute with the differentials. □

The resolution $\{\mathcal{P}, \sigma\} = \{P_n, \sigma_n\}_{n \geq 0}$ appeared in the proof is called the bar resolution of $\underline{R}$.

In general, if $\mathcal{P}$ is an arbitrary projective resolution of $\underline{R}$, then $\mathcal{P}(x)$ is always an exact sequence of $R\,\mathrm{Aut}(x)$-modules beginning with R for any object $x \in \mathrm{Ob}\,\mathcal{C}$, although the modules in $\mathcal{P}(x)$ do not have to be projective. These modules will be projective if $R\,\mathrm{Aut}_{\mathcal{C}}(x)$ is semi-simple, or if $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(y,x)$ for each $y \in \mathrm{Ob}\,\mathcal{C}$ with $\mathrm{Hom}(y,x) \neq \emptyset$ because then each $R\,\mathrm{Hom}_{\mathcal{C}}(y,x)$ is a free $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. For example if $\mathcal{P}$ is the bar resolution of $\underline{R}$ then $\mathcal{P}(x)$ becomes a projective resolution of the $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module R for any $x \in \mathrm{Ob}\,\mathcal{C}$. One might think that when this happens $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M)$ would be related to $\mathrm{Ext}^*_{R\mathcal{C}}(R, M(x))$, at least for certain modules with simple structures. But it is not true as we can see from our first example on cohomology of categories.

Example 3.1.3. *Let $R = \mathbb{F}_2$ be a field of characteristic 2 and $\mathcal{C}$ the following category*



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where $g^2 = 1_x$, $\alpha g = \beta$, $\beta g = \alpha$, $h^2 = 1_y$, $h\alpha = \alpha$ and $h\beta = \beta$. Suppose M is an atomic functor concentrated on y such that $M(y) = \mathbb{F}_2$. Then if we use the bar resolution $\mathcal{P} \to \underline{\mathbb{F}_2} \to 0$ we can get $\mathrm{H}^n(\mathcal{C}; M) \cong \varprojlim^n_{\mathcal{C}} M = 0$ for all $n \geq 0$. On the other hand, the bar resolution $\mathcal{P} \to \underline{\mathbb{F}_2} \to 0$ evaluated at y becomes a projective resolution of $\mathbb{F}_2\,\mathrm{Aut}(y)$-module R, i.e. $\mathcal{P}(y) \to \underline{\mathbb{F}_2}(y) = \mathbb{F}_2 \to 0$, which gives $\mathrm{H}^n(\mathrm{Aut}_{\mathcal{C}}(y), \mathbb{F}_2) = \mathbb{F}_2$ for all $n \geq 0$. We omit the tedious calculations here because after we know more about the projective modules we can compute these groups in a quite neat way, see Example 4.6.11 at the end of Chapter 4 which discusses the same category.

The functor M appeared in the above example is a so-called atomic functor.

Definition 3.1.4. *A functor $M : \mathcal{C} \to R$-mod is called atomic, concentrated on an isomorphism class of objects $[x] \subset \mathrm{Ob}\,\mathcal{C}$ if $M(y)(= 1_y \cdot M) \neq 0$ if and only if $y \cong x$.*

For simplicity, we will just say M is concentrated on x, instead of $[x]$. We also call an $R\mathcal{C}$-module M atomic if the corresponding functor is. The atomic functors are important to us because every functor admits a certain filtration by its subfunctors with the property that every subquotient is an atomic functor. Thus using exact sequences or even spectral sequences consisting of higher limits of subquotients from such a filtration of a functor, one may find the higher limits of the functor. This method for computing higher limits was introduced and studied by Oliver [31], and is frequently used by others, see for instance Grodal [17].

For any $M \in R\mathcal{C}$-mod, $\mathrm{Ext}^*_{R\mathcal{C}}(M, M)$ has a ring structure with the product given by the Yoneda splice, see Benson [4]. When $M = \underline{R}$ the ring is of great importance to us. As a vehicle for studying higher limits, we make the following definition.

Definition 3.1.5. *We call the ring* $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) = \oplus_{i \geq 0} \mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, \underline{R})$ *the cohomology ring of the category algebra $R\mathcal{C}$. The product in this ring is defined by the Yoneda splice of* Ext *classes.*

3.2 Basic properties

We prove some fundamental and important homological properties of category algebras in this section.

Definition 3.2.1. *Suppose $\mu : \mathcal{C}' \to \mathcal{C}$ is a (covariant) functor. We define* $\mathrm{Res}_\mu : (R\text{-}mod)^{\mathcal{C}} \to (R\text{-}mod)^{\mathcal{C}'}$ *to be the restriction along μ. Given a functor $M \in (R\text{-}mod)^{\mathcal{C}}$, we have* $\mathrm{Res}_\mu M = M \cdot \mu \in (R\text{-}mod)^{\mathcal{C}'}$.

Proposition 3.2.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be equivalent small categories. Then*

1. *$R\mathcal{C}$ and $R\mathcal{C}'$ are Morita equivalent, under an equivalence which sends the constant functor to the constant functor; and*

2. *the nerves $N\mathcal{C}$ and $N\mathcal{C}'$ are homotopy equivalent.*

Proof. We prove the first assertion, and the proof for the second can be found in Baues and Wirsching [3].

We want to show the two functor categories $(R\text{-mod})^{\mathcal{C}}$ and $(R\text{-mod})^{\mathcal{C}'}$ are equivalent, because it implies the module categories $R\mathcal{C}$-mod and $R\mathcal{C}'$-mod are equivalent and hence $R\mathcal{C}$ and $R\mathcal{C}'$ are Morita equivalent. In fact if $\mu : \mathcal{C}' \to \mathcal{C}$ and $\nu : \mathcal{C} \to \mathcal{C}'$ are equivalences, we have $\mathrm{Res}_\mu \mathrm{Res}_\nu \cong \mathrm{Id}_{R\mathcal{C}'} : (R\text{-mod})^{\mathcal{C}'} \to (R\text{-mod})^{\mathcal{C}'}$ because of the following diagram

$$\begin{array}{ccc}
M(\nu\mu(x)) = (\mathrm{Res}_\mu \mathrm{Res}_\nu M)(x) & \xrightarrow{\cong} & (\mathrm{Id}_{\mathcal{C}'} M)(x) = M(x) \\
{\scriptstyle M(\nu\mu(\alpha)) = (\mathrm{Res}_\mu \mathrm{Res}_\nu M)(\alpha)} \downarrow & & \downarrow {\scriptstyle (\mathrm{Id}_{\mathcal{C}'} M)(\alpha) = M(\alpha)} \\
M(\nu\mu(y)) = (\mathrm{Res}_\mu \mathrm{Res}_\nu M)(y) & \xrightarrow[\cong]{} & (\mathrm{Id}_{\mathcal{C}'} M)(y) = M(y)
\end{array}$$

where $M \in (R\text{-mod})^{\mathcal{C}'}$, $\alpha : x \to y \in \operatorname{Mor}\mathcal{C}'$ and $\mathrm{Id}_{R\mathcal{C}'}$ is the identity functor. Similarly we can show $\mathrm{Res}_\nu \mathrm{Res}_\mu \cong \mathrm{Id}_{R\mathcal{C}} : (R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{C}}$ so we know Res_μ and Res_ν provide the equivalences between the functor categories. Clearly the constant functor restricts to the constant functor always. □

It's well-known that if a functor $\mu : \mathcal{C}' \to \mathcal{C}$ has a left adjoint $\nu : \mathcal{C} \to \mathcal{C}'$, then Res_ν is also the left adjoint of Res_μ. Under the circumstance, one can show $\mathrm{Ext}^*_{R\mathcal{C}}(\mathrm{Res}_\nu M, N) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(M, \mathrm{Res}_\mu N)$ for any $M \in R\mathcal{C}'$-mod and $N \in R\mathcal{C}$-mod, because both Res_μ and Res_ν are exact (see for example Jackowski-McClure-Oliver[24] II, Proposition 5.1). If μ is indeed an equivalence, we have a stronger result.

Corollary 3.2.3. *Let $\mu : \mathcal{C}' \to \mathcal{C}$ be an equivalence of two small categories. Then we have*

$$\mathrm{Ext}^*_{R\mathcal{C}}(M, N) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(\mathrm{Res}_\mu M, \mathrm{Res}_\mu N),$$

for $M, N \in R\mathcal{C}$-mod. In particular there is a ring isomorphism

$$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(\underline{R}, \underline{R}).$$

Proof. The thing is, $R\mathcal{C}$ and $R\mathcal{C}'$ are Morita equivalent by the functor Res_μ, which takes $\underline{R}$ to $\underline{R}$. □

In [3] Baues and Wirsching showed a special case of the above result. They proved $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(\underline{R}, \mathrm{Res}_\mu N)$ by establishing a homotopy equivalence between the cochain complexes used to define and calculate $\mathrm{H}^*(\mathcal{C}, N)$ and $\mathrm{H}^*(\mathcal{C}', \mathrm{Res}_\mu N)$ (see Proposition 3.1.2). We will discuss a result of Jackowski and Słomińska [25] in the next section, in which they use some "weaker" equivalence between $\mathcal{C}$ and $\mathcal{C}'$ to achieve the reduction $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(\underline{R}, \mathrm{Res}_\mu N)$. Their result will be generalized in Chapter 4 as an application of our theory of vertices and sources.

Corollary 3.2.4. *Given a small category $\mathcal{C}$, we have $\mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong \mathrm{H}^i(|\mathcal{C}|, R)$. Here $|\mathcal{C}|$ stands for the classifying space of $\mathcal{C}$. Following this result we can also show $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong \mathrm{Ext}^*_{R\mathcal{C}'}(\underline{R}, \underline{R})$ as algebras, provided $\mathcal{C}', \mathcal{C}$ are equivalent.*

Proof. By Proposition 3.1.2, we just need to compare $C^n(\mathcal{C}, \underline{R})$ with $\mathrm{Hom}_R(C_n(|\mathcal{C}|), R)$, where $C_n(|\mathcal{C}|)$ is the group of n-chains of $|\mathcal{C}|$. By $\mathrm{Hom}_R(C_n(|\mathcal{C}|), R) \cong \mathrm{Hom}_R(N_n(\mathcal{C}), R)$

and $\mathrm{Hom}_R(N_n(\mathcal{C}), R) \cong C^n(\mathcal{C}, \underline{R})$, we know $C^n(\mathcal{C}, \underline{R})$ and $\mathrm{Hom}_R(C_n(|\mathcal{C}|), R)$ are isomorphic. It's easy to see that the two differentials can be identified with each other. Therefore we have the isomorphism of these cohomology groups.

We may proceed either topologically or algebraically to prove the second part. When $\mathcal{C}$ and $\mathcal{C}'$ are equivalent we know that $|\mathcal{C}|$ and $|\mathcal{C}'|$ are homotopy equivalent by the maps induced by functors between the two categories. So $\mathrm{H}^i(\mathcal{C}, \underline{R}) \cong \mathrm{H}^i(\mathcal{C}', \underline{R})$. Alternatively, because $R\mathcal{C}$ and $R\mathcal{C}'$ are Morita equivalent under an equivalence which preserves the constant functor, the Ext algebras are isomorphic. □

By Proposition 2.3.2 given a category with $\mathrm{Ob}\,\mathcal{C}$ finite, we can identify the functor category $(R\text{-mod})^{\mathcal{C}}$ with the module category $R\mathcal{C}$-mod. Thus given a functor $\mu : \mathcal{C}' \to \mathcal{C}$, the naturally induced functor $\mathrm{Res}_\mu : (R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{C}'}$, can be considered as $\mathrm{Res}_\mu : R\mathcal{C}\text{-mod} \to R\mathcal{C}'\text{-mod}$. On the other hand, since $\mu : \mathcal{C}' \to \mathcal{C}$ extends linearly to a natural map of R-modules $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$, it's reasonable to ask if $\bar{\mu}$ is an algebraic homomorphism and if it is, whether or not its induced functor, the so-called change-of-ring or restriction, $\downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$: $R\mathcal{C}$-mod $\to R\mathcal{C}'$-mod, coincides with Res_μ. The answer to the first question is no, and here is a simple example. Let $\mathcal{C}'$ be a category with two objects and only identity maps, and let $\mathcal{C}$ be a category with one object and the identity map along with the unique functor $\mu : \mathcal{C}' \to \mathcal{C}$. Then the map $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$ is not an algebra homomorphism for the product of the two morphisms in $\mathcal{C}'$ is zero while the product of their images is not. We show when μ will be an algebra homomorphism.

Proposition 3.2.5. *A functor $\mu : \mathcal{C}' \to \mathcal{C}$ extends linearly to an algebra homomorphism $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$ if and only if μ is injective on* $\mathrm{Ob}\,\mathcal{C}'$. *When this happens, the induced functor followed by $1_{\mathcal{C}'}$, $1_{\mathcal{C}'} \cdot \downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$: $R\mathcal{C}$-mod $\to R\mathcal{C}'$-mod is exactly* Res_μ.

Proof. We know $\mu(\beta\alpha) = \mu(\beta)\mu(\alpha)$ for any pair of composable morphisms α, β in $\mathcal{C}'$. The injectivity of μ implies two morphisms $\alpha, \beta \in \mathrm{Mor}(\mathcal{C}')$ are composable if and only if $\mu(\alpha), \mu(\beta) \in \mathrm{Mor}(C)$ are composable.

If μ is injective on $\mathrm{Ob}\,\mathcal{C}'$, then we define a map $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$ as the linear extension of functor μ, i.e., $\bar{\mu}(\sum_i r_i\alpha_i) = \sum_i r_i\bar{\mu}(\alpha_i)$ for any $r_i \in R, \alpha_i \in \mathrm{Mor}(\mathcal{C}')$. This $\bar{\mu}$ is indeed an algebra homomorphism because our previous observation of μ implies $\bar{\mu}((\sum_j r_j\beta_j)(\sum_i r_i\alpha_i)) = \bar{\mu}(\sum_j r_j\beta_j)\bar{\mu}(\sum_i r_i\alpha_i)$ is always true.

On the other hand if the linear extension $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$ is an algebra homomorphism then we must have $\bar{\mu}(0) = 0$ and then $\bar{\mu}(1_x)\bar{\mu}(1_y) = \bar{\mu}(1_x \cdot 1_y) = 0$ unless $x = y$. This suggests that μ is injective on $\mathrm{Ob}\,\mathcal{C}'$.

When $\bar{\mu} : R\mathcal{C}' \to R\mathcal{C}$ is an algebra homomorphism, we show $1_{\mathcal{C}'} \cdot \downarrow^{R\mathcal{C}}_{R\mathcal{C}'} = \mathrm{Res}_\mu : R\mathcal{C}$-mod $\to R\mathcal{C}'$-mod. Let M be an $R\mathcal{C}$-module and $\gamma \in \mathrm{Mor}(\mathcal{C}')$. Then $M = M_M$ for some $M \in (R\text{-mod})^{\mathcal{C}}$. The $R\mathcal{C}'$-module structures of $M = M_M$ given by Res_μ and $1_{\mathcal{C}'} \cdot \downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$ coincide for $\mathrm{Res}_\mu M_M = M_{\mathrm{Res}_\mu M}$, $\gamma \cdot M_M = \mu(\gamma) M_M$ and $1_{\mathcal{C}'}$ kills the elements of $M \downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$ that aren't supported on any objects of $\mathcal{C}'$. □

In Chapter 4, we will take $\mathcal{C}'$ as a full subcategory of $\mathcal{C}$. Then the restriction $\mathrm{Res}_\iota : R\mathcal{C}$-mod $\to R\mathcal{C}'$-mod is determined by the algebra homomorphism $\bar{\iota} : R\mathcal{C}' \to R\mathcal{C}$, hence by $\iota : \mathcal{C}' \to \mathcal{C}$. For this reason we won't distinguish Res_ι and $\downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$ from now on, and will write $\downarrow^{R\mathcal{C}}_{R\mathcal{C}'}$ and Res_ι as $\downarrow^{\mathcal{C}}_{\mathcal{C}'}$ which is common in representation theory.

There is another cohomology theory we can apply to category algebras, namely the Hochschild cohomology. In general this is not the same as $\mathrm{H}^i(\mathcal{C}, \underline{R})$ and examples where the two cohomology groups are different can be found when $\mathcal{C}$ is a group (Benson [5], Theorem 2.11.2). However, when $\mathcal{C}$ is a poset these cohomology theories do coincide, and this is a theorem of Gerstenhaber and Schack [16]. One can find a description of the Hochschild cohomology theory in [16], as well as in Mac Lane [30] (X.3) and Happel [19].

Theorem 3.2.6. *[16] Let Γ be a poset and R a commutative ring with identity. Let $|\Gamma|$ be the topological realization, i.e., the order complex, of Γ. Then $\mathrm{H}^i(|\Gamma|, R) \cong \mathrm{HH}^i(R\Gamma, R\Gamma)$, where $R\Gamma$ is the incidence algebra.*

If Γ, Γ' are equivalent as categories, then from a standard result in the Hochschild cohomology theory we have $\mathrm{HH}^*(R\Gamma, R\Gamma) \cong \mathrm{HH}^*(R\Gamma', R\Gamma')$ since $R\Gamma$ and $R\Gamma'$ are Morita equivalent. Hence $\mathrm{H}^*(|\Gamma|, R) \cong \mathrm{H}^*(|\Gamma'|, R)$ which matches the statements of Corollaries 3.2.3 and 3.2.5.

3.3 Overcategories and a reduction theorem

We have seen in last section that a certain pair of adjoint functors μ, ν between $\mathcal{D}$ and $\mathcal{C}$ leads to a reduction $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, \mathrm{Res}_\mu N)$ because of the existence of another pair of (induced) adjoint functors $\mathrm{Res}_\mu, \mathrm{Res}_\nu$ between the corresponding functor categories. The truth is, a single functor $\mu : \mathcal{D} \to \mathcal{C}$ is enough to induce a pair of adjoint functors on the level of functor categories. This time the left adjoint of Res_μ is called the left Kan extension of μ, and is not necessarily induced by any functor from $\mathcal{C}$ to $\mathcal{D}$. Whenever there exists a pair of adjoint functors on functor categories, one may find suitable conditions on the triple $\iota : \mathcal{D} \to \mathcal{C}$ so that one can establish isomorphisms between the cohomology groups $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N)$ and $\mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, \mathrm{Res}_\iota N)$ (or something similar). This consideration leads us to an important reduction theorem, Corollary 3.2.3, which is due to Jackowski and Słomińska [25].

First of all, let's review the definition of the left Kan extension, which uses the so-called overcategories $\iota \downarrow_y, y \in \mathrm{Ob}\,\mathcal{C}$.

Definition 3.3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories along with a functor $\iota : \mathcal{D} \to \mathcal{C}$. For each $y \in \mathrm{Ob}\,\mathcal{C}$, the overcategory $\iota \downarrow_y$ consists of objects (x, α), where $x \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\iota(x), y)$. The morphisms in the overcategory are determined via completing commutative diagrams as follows*

$$\begin{array}{ccc} \iota(x) & & \\ \uparrow{\scriptstyle \iota(\beta)} & \searrow^{\alpha} & \\ & & y, \\ & \nearrow_{\alpha'} & \\ \iota(x') & & \end{array}$$

where $(x, \alpha), (x', \alpha') \in \mathrm{Ob}\,\iota \downarrow_y$ and $\beta \in \mathrm{Hom}_{\mathcal{D}}(x, x')$.

When $\mathcal{C}$ is a poset and $\mathcal{D}$ is a subposet along with the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$, the overcategory $\iota \downarrow_y$ is isomorphic in a natural way to $\mathcal{D}_{\leq y}$, a subposet of $\mathcal{D}$ consisting of all $x \in \mathrm{Ob}\,\mathcal{D}$ which satisfy the condition that $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$.

It's easy to see that for a fixed functor $\iota : \mathcal{D} \to \mathcal{C}$ between two small categories, we can naturally define two functors $\iota \downarrow_? : \mathcal{C} \to sCat$, the category of small categories, and $|\iota \downarrow_?| : \mathcal{C} \to Spaces$, the category of topological spaces. For any $y \in \mathrm{Ob}\,\mathcal{C}$, we use $C_*(\iota \downarrow_y)$ to denote the simplicial complex coming from the simplicial set associated to the small category $\iota \downarrow_y$, which can be regarded as the chain complex of the space $|\iota \downarrow_y|$.

Definition 3.3.2. *The left Kan extension of a functor* $\iota : \mathcal{D} \to \mathcal{C}$ *is defined by*

$$K(M)(y) = \varinjlim_{\iota\downarrow_y} M \circ \pi,$$

where $M \in R\mathcal{D}$*-mod,* $y \in \operatorname{Ob}\mathcal{C}$ *and* $\pi : \iota \downarrow_y \to \mathcal{D}$ *is the projection* $(x, \alpha) \mapsto x$*.*

Since K is the left adjoint of the exact functor Res_ι, it preserves projective modules. It is not exact in general, but when it is it sends any projective $R\mathcal{D}$-resolution to a projective $R\mathcal{C}$-resolution.

In Proposition 3.1.2, we have seen an explicit way to construct, for an arbitrary small category $\mathcal{D}$, the bar resolution $\mathcal{P}$ of $\underline{R} = P_{-1}$. In fact it's not hard to see that the exact sequence $\mathcal{P} \to \underline{R} \to 0$ evaluated at each $x \in \operatorname{Ob}\mathcal{D}$, i.e. $\mathcal{P}(x) \to R \to 0$, is isomorphic to the reduced simplicial chain complex of the classifying space $|\,\mathrm{Id}_{\mathcal{D}} \downarrow_x |$, where $\mathrm{Id}_{\mathcal{D}} \downarrow_x$ is constructed via the identity functor $\mathrm{Id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, see Gabriel-Zisman [14] and Grodal [17]. Since every $\mathrm{Id}_{\mathcal{D}} \downarrow_x$ has a terminal object $(x, 1_x)$, by Quillen's Theorem A [33], the space $|\,\mathrm{Id}_{\mathcal{D}} \downarrow_x |$ is contractible and hence we get another proof of the exactness of the bar resolution $\mathcal{P} \to \underline{R} \to 0$. It was observed by Dwyer and Kan [11] that $K(\mathcal{P}) \cong K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?)) \to K(\underline{R}) \to 0$ is isomorphic to the complex of $R\mathcal{C}$-modules $C_*(\iota \downarrow_?) \to K(\underline{R}) \to 0$, in which every $C_*(\iota \downarrow_?) \cong K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?))$ remains projective. When $K(\underline{R})$ happens to be isomorphic to $\underline{R}$ the complex evaluated at any $y \in \operatorname{Ob}\mathcal{C}$ becomes the reduced simplicial complex $C_*(\iota \downarrow_y) \to R \to 0$ of $|\iota \downarrow_y |$. It was shown by Jackowski and Słomińska [25] that this complex of $R\mathcal{C}$-modules $C_*(\iota \downarrow_?) \to K(\underline{R}) \to 0$ will become a projective resolution of the trivial $R\mathcal{C}$-module $\underline{R}$ if we assume that all overcategories associated to $\iota : \mathcal{D} \to \mathcal{C}$ are R-acyclic (a small category is said to be R-acyclic if all reduced homology groups of its classifying space with coefficients in R vanish).

When we turn to our representation-theoretic settings, the left adjoint of the restriction $\mathrm{Res}_\iota : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ is normally written as the induction $\uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} -$, provided $R\mathcal{C}$ is a right $R\mathcal{D}$-module. By Proposition 3.2.6, $R\mathcal{C}$ becomes an $R\mathcal{D}$-module if $\iota : \mathcal{D} \to \mathcal{C}$ is injective on the object set. When this happens we must have $K \cong \uparrow_{\mathcal{D}}^{\mathcal{C}}$ as two functors, and particularly must have $K(\mathcal{P}_{\mathcal{D}}) \cong \mathcal{P}\uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P}$. Obviously the complex $K(\mathcal{P})$ will be exact if $R\mathcal{C}$ is a flat $R\mathcal{D}$-module. Let's put these important observations into the following lemma. The first statement in the lemma was known to Dwyer and Kan [11], and the second is new, though a special form of it has appeared in Symonds [37]. The conclusion is essentially obtained by Jackowski and Słomińska [25].

Lemma 3.3.3. *Let $\iota : \mathcal{D} \to \mathcal{C}$ be a functor and $\mathcal{P}$ be the bar resolution of the $R\mathcal{D}$-module $\underline{R}$. Then*

1. *there are isomorphisms of complexes of projective $R\mathcal{C}$-modules*

$$K(\mathcal{P}) \cong K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?)) \cong C_*(\iota \downarrow_?).$$

   For each $y \in \mathrm{Ob}\,\mathcal{C}$, $K(\mathcal{P})(y)$ is isomorphic to the simplicial chain complex of $\iota \downarrow_y$. If furthermore $\iota : \mathcal{D} \to \mathcal{C}$ is injective on objects, then the above three complexes are isomorphic to $R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P}$; and

2. *if every $\iota \downarrow_y$ is non-empty and connected, then $K(\underline{R}) \cong \underline{R}$.*

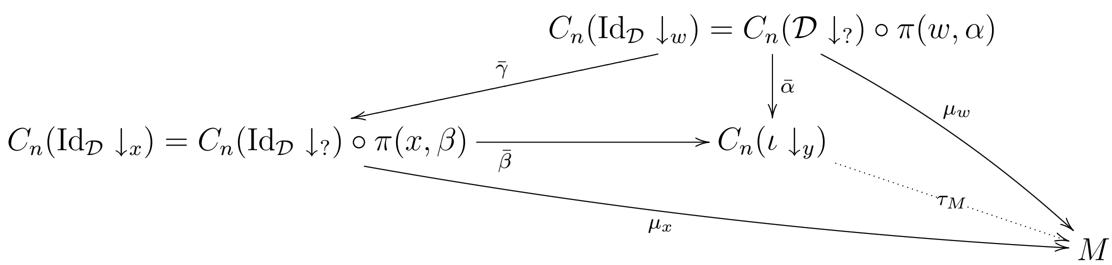
Thus if for every $y \in \mathrm{Ob}\,\mathcal{C}$ the overcategory $\iota \downarrow_y$ is R-acyclic, we obtain a projective resolution

$$K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?)) \cong C_*(\iota \downarrow_?) \to K(\underline{R}) \cong \underline{R} \to 0$$

of the trivial $R\mathcal{C}$-module $\underline{R}$, which, evaluated at y, is isomorphic to the reduced simplicial chain complex of $|\iota \downarrow_y|$.

Proof. Part of the proof to (1) is more or less explained in the two paragraphs after Definition 3.3.2. In order to show $K(\mathcal{P}) \cong K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?))$, we just need to prove $\mathcal{P} \cong C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?)$, which is clear if we write out these two complexes and their evaluations at each object of $\mathcal{C}$.

Let y be an object of $\mathcal{C}$. We can show for each $n \geq 0$ that $C_n(\iota \downarrow_y)$ is isomorphic to $\varinjlim_{\iota\downarrow_y} C_n(\mathrm{Id}_{\mathcal{D}} \downarrow_?) \circ \pi$, by the universal property of $\varinjlim$. Let $(w, \alpha) \in \mathrm{Ob}\,\mathrm{Id}_{\mathcal{D}} \downarrow_y$, $(x, \beta) \in \mathrm{Ob}\,\mathrm{Id}_{\mathcal{D}} \downarrow_y$ and $\gamma \in \mathrm{Hom}((w, \alpha), (x, \beta))$, i.e., $\alpha = \beta\iota(\gamma)$. Then we have the following commutative diagram



where $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ denote the composite by α, β and γ, respectively. Let M be another R-module that fits into the commutative diagram. We want to construct a map τ_M from $C_n(\iota\downarrow_y)$ to M so that we can say $C_n(\iota\downarrow_y)$ is isomorphic to $\varinjlim_{\iota\downarrow_y} C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_?)\circ\pi$. In fact, if for any n-chain in $C_n(\iota\downarrow_y)$ ending with (w,α), say

$$(w_0,\alpha_0)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1})\xrightarrow{\gamma_{n-1}}(w,\alpha),\cdots\cdots\cdots(*)$$

we can find a preimage c of it in some $C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_z), z\in\mathrm{Ob}\,\mathcal{D}$, then we define a map τ_M from $C_n(\iota\downarrow_y)$ to M by $\tau_M=\mu_z(c)$. If we can show this τ_M is well-defined and unique, then by the universal property of $\varinjlim$, we get an isomorphism

$$C_n(\iota\downarrow_y)\cong\varinjlim_{\mathrm{Id}_{\mathcal{D}}\downarrow_y} C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_?)\circ\pi.$$

Now we verify the validity of τ_M. First of all, there exists a preimage of such an element $(w_0,\alpha_0)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1})\xrightarrow{\gamma_{n-1}}(w,\alpha)$, because we can always construct from it an element of $C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_w)$

$$c=(w_0,\alpha_0/\alpha)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1}/\alpha)\xrightarrow{\gamma_{n-1}}(w,1_w),$$

where α_i/α denotes $\gamma_{n-1}\cdots\gamma_i$ since $\alpha_i=\alpha\iota(\gamma_{n-1})\cdots\iota(\gamma_i)$. It's easy to verify this is a preimage of $(*)$. Second of all, we shall prove the definition of τ_M is independent of the choice of the preimage of $(*)$. The thing is, any preimage of $(w_0,\alpha_0)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1})\xrightarrow{\gamma_{n-1}}(w,\alpha)$ must lie in some $C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_z)$ and must be in the image of some map from $C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_w)$ to $C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_z)$. Suppose $c'=(v_0,\theta_0)\xrightarrow{\sigma_0}\cdots\to(v_{n-1},\theta_{n-1})\xrightarrow{\sigma_{n-1}}(v,\theta)$ is a preimage of the given $(*)$, where $\theta_i,\theta\in\mathrm{Hom}_{\mathcal{D}}(v,z)$. Then there exists some $\eta\in\mathrm{Hom}_{\mathcal{C}}(\iota(z),y)$ inducing a map $\bar{\eta}:C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_z)\to C_n(\iota\downarrow_y)$, which sends c' to $(*)$. Thus from

$$\begin{aligned}&\bar{\eta}[(v_0,\theta_0)\xrightarrow{\sigma_0}\cdots\to(v_{n-1},\theta_{n-1})\xrightarrow{\sigma_{n-1}}(v,\theta)]\\ =\;&(v_0,\eta\iota(\theta_0))\xrightarrow{\sigma_0}\cdots\to(v_{n-1},\eta\iota(\theta_{n-1}))\xrightarrow{\sigma_{n-1}}(v,\eta\iota(\theta))\\ =\;&(w_0,\alpha_0)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1})\xrightarrow{\gamma_{n-1}}(w,\alpha),\end{aligned}$$

it must be true that $v=w,\eta\iota(\theta)=\alpha$ and $v_i=w_i$, $\gamma_i=\sigma_i$, $\eta\iota(\theta_i)=\alpha_i$ for all i. Since $\theta_i=\theta\sigma_{n-1}\cdots\sigma_i$ and $\alpha_i=\alpha\iota(\gamma_{n-1})\cdots\iota(\gamma_i)$, these imply $\theta(c)=c'$ and $\mu_z(c')=\mu_z(\bar{\theta}(c))=\mu_w(c)$ in the commutative diagram, where

$$c=(w_0,\alpha_0/\alpha)\xrightarrow{\gamma_0}\cdots\to(w_{n-1},\alpha_{n-1}/\alpha)\xrightarrow{\gamma_{n-1}}(w,1_w).$$

Note that $\alpha_i/\alpha = \gamma_{n-1}\cdots\gamma_i$. Hence the map τ_M is well defined. Using once again the universal property and similar arguments as above we can extend isomorphism of the form $\varinjlim_{\mathrm{Id}_{\mathcal{D}}\downarrow_y} C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_?)\circ\pi = K(C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_?))(y) \cong C_n(\iota\downarrow_y)$ to an isomorphism of functors $K(C_n(\mathrm{Id}_{\mathcal{D}}\downarrow_?)) \cong C_n(\iota\downarrow_?)$.

In the end, let $\iota : \mathcal{D} \to \mathcal{C}$ be injective on $\mathrm{Ob}\,\mathcal{D}$. Then $R\mathcal{C}\otimes_{R\mathcal{D}} -$ is well-defined and hence $K \cong R\mathcal{C}\otimes_{R\mathcal{D}} -$. So we have all isomorphisms of (1).

Statement (2) follows directly from the definition of Kan extension. □

Note that if all the overcategories $\iota\downarrow_y$, $y\in\mathrm{Ob}\,\mathcal{C}$, are contractible, then $|\mathcal{D}|\simeq|\mathcal{C}|$ by Quillen's Theorem A. As a consequence, we get $\mathrm{H}^*(\mathcal{C},\underline{R}) \cong \mathrm{H}^*(\mathcal{D},\underline{R})$. This is a special case of the following reduction theorem on higher limits, which also generalizes Corollary 3.2.3.

Corollary 3.3.4. *(Jackowski-Słomińska [25]) Let $\iota : \mathcal{D}\to\mathcal{C}$ satisfy the condition that every $\iota\downarrow_y, y\in\mathrm{Ob}\,\mathcal{C}$, is R-acyclic. Then $\varprojlim^*_{\mathcal{C}} M \cong \varprojlim^*_{\mathcal{D}}\mathrm{Res}_\iota M$ for any $R\mathcal{C}$-module M. In particular, when $R\mathcal{C}$ is a right projective $R\mathcal{D}$-module and $\underline{R}\uparrow_{\mathcal{D}}^{\mathcal{C}} \cong \underline{R}$ these isomorphisms exist.*

Proof. We take the bar resolution $\mathcal{P}$ of $\underline{R}$. By the previous lemma, its left Kan extension $K(\mathcal{P})$ is a projective resolution of the $R\mathcal{C}$-module $\underline{R}$. So we have for an arbitrary $R\mathcal{C}$-module M

$$\mathrm{Hom}_{R\mathcal{C}}(K(\mathcal{P}),M) \cong \mathrm{Hom}_{R\mathcal{D}}(\mathcal{P},\mathrm{Res}_\iota M),$$

by the adjunction of these two functors K and Res_ι. Since the cochain complex $\{\mathrm{Hom}_{R\mathcal{C}}(K(\mathcal{P}),M)\}$ computes $\varprojlim^*_{\mathcal{C}} M$ and the other cochain complex $\{\mathrm{Hom}_{\mathcal{D}}(\mathcal{P},M)\}$ calculates $\varprojlim^*_{\mathcal{D}}\mathrm{Res}_\iota M$, the isomorphisms between the higher limits over $\mathcal{C}$ and $\mathcal{D}$ are established. □

Since the isomorphism in Corollary 3.3.4 can be written in terms of Ext groups $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R},N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R},\mathrm{Res}_\iota N)$, it reminds us of the Eckmann-Shapiro Lemma in the cohomology theory of algebras, which compares Ext groups after a base ring change (Benson [5]). There are two different types of isomorphisms given by the Eckmann-Shapiro Lemma, one established using the induction and the other using the co-induction. We record them below as Lemmas 3.3.5 and 3.3.6.

Lemma 3.3.5. *(Eckmann-Shapiro) Let M be an arbitrary $R\mathcal{C}$-module. If $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module then for any $R\mathcal{C}$-module N, we have*

$$\mathrm{Ext}^*_{R\mathcal{C}}(M\uparrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M, N\downarrow^{\mathcal{C}}_{\mathcal{D}}).$$

Proof. Suppose $\mathcal{P} \to M \to 0$ is a projective $R\mathcal{D}$-resolution. Then $\mathcal{P}\uparrow^{\mathcal{C}}_{\mathcal{D}} \to M\uparrow^{\mathcal{C}}_{\mathcal{D}} \to 0$ is also a projective resolution since $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module and $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projectives. The Ext groups $\mathrm{Ext}^*_{R\mathcal{D}}(M, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$ are the homology groups of $\mathrm{Hom}_{R\mathcal{D}}(\mathcal{P}, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$, which is isomorphic to $\mathrm{Hom}_{R\mathcal{C}}(\mathcal{P}\uparrow^{\mathcal{C}}_{\mathcal{D}}, N)$. Since the latter gives $\mathrm{Ext}^*_{R\mathcal{C}}(M\uparrow^{\mathcal{C}}_{\mathcal{D}}, N)$, it follows that

$$\mathrm{Ext}^*_{R\mathcal{C}}(M\uparrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M, N\downarrow^{\mathcal{C}}_{\mathcal{D}}).$$

□

If we replace M by $\underline{R}$ in the Eckmann-Shapiro Lemma, we get $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}\uparrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$, something slightly different from the isomorphism of Jackowski and Słomińska: $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$. The isomorphism $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}\uparrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\underline{R}, N\downarrow^{\mathcal{C}}_{\mathcal{D}})$ doesn't give a reduction for computing higher limits, unless $\underline{R} \cong \underline{R}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ where the first $\underline{R}$ is an $R\mathcal{C}$-module while the second is an $R\mathcal{D}$-module. We may readily ask if it's possible to obtain $\underline{R} \cong \underline{R}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ for some $\mathcal{D} \subset \mathcal{C}$, and this is indeed the reason why we look into the theory of relative projectivity and is our motivation for developing the theory of vertices and sources in Chapter 4.

When $R\mathcal{C}$ is projective as a left $R\mathcal{D}$-module, using the right adjoint $\Uparrow^{\mathcal{C}}_{\mathcal{D}}$ of $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ we can get a different formula, which is less useful than Lemma 3.3.6 in our situation. Recall that the co-induction of M is defined by $M\Uparrow^{\mathcal{C}}_{\mathcal{D}} = \mathrm{Hom}_{R\mathcal{D}}(R\mathcal{C}, M)$. The proof of the following lemma is similar to the previous one.

Lemma 3.3.6. *(Eckmann-Shapiro) Suppose $R\mathcal{C}$ is a projective $R\mathcal{D}$-module. Then for any $R\mathcal{C}$-module M and $R\mathcal{D}$-module N we have* $\mathrm{Ext}^*_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Ext}^*_{R\mathcal{C}}(M, N\Uparrow^{\mathcal{C}}_{\mathcal{D}})$.

In transition to the main chapter of this thesis, we would like to make a comment on the proof of Corollary 3.3.4, which complements the comment after Lemma 3.3.5 and gives us some idea about what we do next. The essence of that proof is to use the isomorphism of complexes $\mathrm{Hom}_{R\mathcal{C}}(K(\mathcal{P}), M) \cong \mathrm{Hom}_{R\mathcal{D}}(\mathcal{P}, \mathrm{Res}_\iota M)$. In order to pass from this isomorphism to those between higher limits, we need to have the left Kan extension of $\mathcal{P}$, $K(\mathcal{P})(\cong R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P})$ to be exact. Plus we want $K(\underline{R})(\cong R\mathcal{C} \otimes_{R\mathcal{D}} \underline{R}) \cong \underline{R}$ so we do have a projective resolution for $\underline{R}$ (see comment after Lemma 3.3.6). In the original proof of Jackowski and Słomińska, the R-acyclicity of $\iota \downarrow_y, y \in \mathrm{Ob}\,\mathcal{C}$ resolves both issues we just mentioned. However, in practice it's hard to detect the R-acyclicity of the overcategories $\iota \downarrow_y, y \in \mathrm{Ob}\,\mathcal{C}$ because of their complexity. Hence we turn to consider the induction $\uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} -$, which is isomorphic to K. The reasons are that firstly the flatness of $R\mathcal{C}$ will result in the exactness of K (hence the exactness of $K(C_*(\mathrm{Id}_{\mathcal{D}} \downarrow_?)))$, and secondly it's relatively easier to formulate conditions on $\mathcal{D}$ to the conditions that $R\mathcal{C}$ becomes flat (see Lemma 1.3 of [6] for an early example). This naturally amounts to studying the flatness (in practice, the projectivity) of $R\mathcal{C}$ as an $R\mathcal{D}$-module. In the next chapter, Section 4.2 deals with the projectivity of $R\mathcal{C}$, and Section 4.4 tells us the existence of $\underline{R} \cong \underline{R}\uparrow_{\mathcal{D}}^{\mathcal{C}}$.

In the end we give an example where all overcategories are R-acyclic, while $R\mathcal{C}$ is not a right flat $R\mathcal{D}$-module.

Example 3.3.7. *Let $\mathcal{C}$ be a poset and $\mathcal{D}$ be a subposet as follows.*

It's easy to check all the overcategories are R-acyclic. In fact, $\iota \downarrow_x, \iota \downarrow_y$ and $\iota \downarrow_w$ are contractible because each of them has a terminal object. The other one, $\iota \downarrow_z$, has the reduced simplicial complex

$$0 \to R\{\alpha, \beta\} \xrightarrow{\delta} R\{1_x, 1_y, 1_w\} \xrightarrow{\epsilon} R \to 0,$$

whose homology groups are zeros. Here ϵ is the augmentation map and δ sends α and β to $1_x - 1_w$ and $1_y - 1_w$, respectively.

On the other hand, as a right $R\mathcal{D}$-module, $R\mathcal{C}$ is a direct sum

$$R\mathcal{C} = R\{1_w\} \oplus R\{1_x, \alpha\} \oplus R\{1_y, \beta\} \oplus R\{1_z\} \oplus R\{\gamma, \mu, \gamma\alpha = \mu\beta\}.$$

The first 3 summands are right projective modules (see Section 4.1) which add up to $R\mathcal{D}$, the fourth is a null module on which $R\mathcal{D}$ acts as zero and the last is isomorphic to the constant module $\underline{R}$ which isn't flat as a right $R\mathcal{D}$-module.

However if we take $\mathcal{E} = \{w\} \subset \mathcal{C}$, then $R\mathcal{C}$ becomes a right projective $R\mathcal{E}$-module, and meanwhile every overcategory associated with $\iota : \mathcal{E} \to \mathcal{C}$ is R-acyclic.

Chapter 4

EI-categories, relative projectivity, vertices and sources

In this chapter, we investigate the representation theory of EI-categories and its applications to cohomology theory, especially to the computation of higher limits. We assume the base ring R is a field or a complete discrete valuation ring, in order to have the unique decomposition property for every $R\mathcal{C}$-module. When R is a field of characteristic $p > 0$, we denote it by $\mathbb{F}_p$ (instead of $\mathbb{F}_q$ for $q = p^n$), and require it to be large enough (e.g. algebraically closed etc.) if necessary.

Definition *An EI-category is a small category $\mathcal{C}$ in which all endomorphisms are isomorphisms.*

Posets and groups are EI-categories. Recently some very interesting and important EI-categories have appeared in the representation theory of groups and the homotopy theory of classifying spaces of groups. We have in mind the Frobenius, Brauer and Puig categories described in the book by Thévenaz ([38] Chap.7), the orbit categories and other categories considered by Dwyer [10], and the fusion systems of Puig, used by Broto, Levi and Oliver [6].

Some of the general theory of EI-categories is given by tom Dieck in [9] (I.11), much of which was due to Lück, see also [26]. One of the important features of EI-categories is described as follows. Given an EI-category $\mathcal{C}$, there is a preorder defined on $\mathrm{Ob}\,\mathcal{C}$, that is $x \leq y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x,y) \neq \emptyset$. Let $[x]$ be the isomorphism class of an object $x \in \mathrm{Ob}\,\mathcal{C}$. This preorder induces a partial order on the set $\mathrm{Is}\,\mathcal{C}$ of isomorphism classes of $\mathrm{Ob}\,\mathcal{C}$ (specified by $[x] \leq [y]$ if and only if $\mathrm{Hom}_{\mathcal{C}}(x,y) \neq \emptyset$), which plays an important role in studying representations and cohomology of EI-categories. Because of the existence of an order for the isomorphism classes of objects in any EI-category, EI-categories are sometimes referred as "ordered categories" by some authors, see Oliver [31] and Jackowski and Słomińska [25]. The preorder makes it possible to construct the following useful full subcategories. For an EI-category $\mathcal{C}$ and an object $x \in \mathrm{Ob}\,\mathcal{C}$, we define a full subcategory $\mathcal{D}_{\leq x} \subset \mathcal{D}$, consisting of all $y \in \mathrm{Ob}\,\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{C}}(y,x) \neq \emptyset$. Similarly we can define several other full subcategories of $\mathcal{D}$: $\mathcal{D}_{<x}, \mathcal{D}_{\geq x}$ and $\mathcal{D}_{>x}$.

Convention In the rest of this thesis, we're going to assume that $\mathcal{C}$ is finite (i.e. $\mathrm{Mor}(\mathcal{C})$ is finite). Then $R\mathcal{C}$ becomes a finite dimensional algebra and $1_{R\mathcal{C}}$ has a primitive decomposition.

When we talk about a full subcategory $\mathcal{D}$ of an EI-category $\mathcal{C}$, we suppose $\mathcal{D}$ has the following property: if $x \in \mathrm{Ob}\,\mathcal{D}$, then $[x] \subset \mathrm{Ob}\,\mathcal{D}$, where $[x]$ is the isomorphism class of x in $\mathcal{C}$.

Let $\mathcal{C}$ be an EI-category and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. The above condition on $\mathcal{D}$ is a natural requirement, which won't change the nature of any questions to be considered here and does save us from some unnecessary technical concerns. The thing is, if we are to investigate an $R\mathcal{C}$-module M, then (as a functor) M has to take an isomorphic value on every object of an isomorphism class of objects in $\mathcal{C}$. If $\mathcal{D}$ doesn't meet our requirement, then we can construct a full subcategory $\mathcal{E} \subset \mathcal{C}$, which contains $\mathcal{D}$ as a full subcategory and is equivalent to $\mathcal{D}$. In fact, we let $\mathrm{Ob}\,\mathcal{E}$ consist of all objects in $\mathcal{C}$ which are isomorphic to some object(s) in $\mathcal{D}$, and then the structure of $\mathcal{E}$ becomes clear. Since the representation and cohomology theories of $\mathcal{E}$ is the same as those of $\mathcal{D}$ and $N \uparrow_{\mathcal{D}}^{\mathcal{C}} \cong (N \uparrow_{\mathcal{D}}^{\mathcal{E}}) \uparrow_{\mathcal{E}}^{\mathcal{C}}$ for any $N \in R\mathcal{D}$-mod, we can study the connections between representation and cohomology theories of $R\mathcal{C}$-mod and those of $R\mathcal{E}$-mod, in order to understand the connections between $R\mathcal{C}$-mod and $R\mathcal{D}$-mod.

In Section 4.1, we describe the structures of the simple and indecomposable projective $R\mathcal{C}$-modules. Using the structures of indecomposable projective modules, we can generalize a result of Symonds [37] on the projectivity of the trivial module $\underline{R}$. Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. We show in Section 4.2 when a projective $R\mathcal{C}$-module, regarded as an $R\mathcal{D}$-module, remains projective. We briefly go over the general theory of relative projectivity of $R\mathcal{C}$-modules in Section 4.3, where we also give an example to explain how it works with the Eckmann-Shapiro Lemma to calculate the Ext groups over $\mathcal{C}$. Our main results accumulate in Section 4.4, in which we develop the theory of vertices and sources. In this section we associate to each indecomposable $R\mathcal{C}$-module M a vertex, which is the smallest full convex subcategory of $\mathcal{C}$, relative to which M is projective. This implies every indecomposable $R\mathcal{C}$-module is an induced module from some subcategory (e.g. its vertex). Since we are particularly interested in the calculation of higher limits, we will describe in Section 4.5 the structure of the vertex of the trivial $R\mathcal{C}$-module $\underline{R}$. We end this chapter with discussions of finite categories with subobjects. We show that some important topological properties of such a category are determined by those of its underlying poset.

4.1 Projective modules and simple modules

Most of the results in this section was obtained by Lück [26]. An account of his work can also be found in tom Dieck [9]. Remember we assume the base ring R is a field or a complete discrete valuation ring.

Let $\mathcal{C}$ be a small category and $x \in \mathrm{Ob}\,\mathcal{C}$ an object. Suppose P_x is a projective $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. Then the induced module $R\mathcal{C} \otimes_{R\,\mathrm{Aut}(x)} P_x$ is a projective $R\mathcal{C}$-module, because $R\mathcal{C} \otimes_{R\,\mathrm{Aut}(x)} R\,\mathrm{Aut}(x) \cong R\mathcal{C}$ is projective and P_x is isomorphic to a direct summand of $R\,\mathrm{Aut}(x)^n$ for some positive integer n. If $\mathcal{C}$ is a finite EI-category, we show the list of projective modules of the form: $R\mathcal{C} \otimes_{R\,\mathrm{Aut}_{\mathcal{C}}(x)} P$ for some $x \in \mathrm{Ob}\,\mathcal{C}$ and an indecomposable projective $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module P, exhausts all indecomposable projective $R\mathcal{C}$-modules.

In order to achieve the goal, we analyze the structures of the regular $R\mathcal{C}$-module and its indecomposable summands. Suppose $\mathcal{C}$ is a finite EI-category. It's easy to see that the regular module of the category algebra $R\mathcal{C}$ decomposes into a direct sum $\oplus_{x\in \mathrm{Ob}\,\mathcal{C}} R\mathcal{C} \cdot 1_x$, where 1_x is the identity of $\mathrm{Aut}_{\mathcal{C}}(x)$. If $x \cong x'$ are two isomorphic objects, and $f \in \mathrm{Hom}_{\mathcal{C}}(x, x')$ is an isomorphism, then the assignment $\alpha \mapsto \alpha \cdot f$ for each $\alpha \in R\mathcal{C} \cdot e$ defines an isomorphism of $R\mathcal{C}$-modules $R\mathcal{C} \cdot 1_{x'} \to R\mathcal{C} \cdot 1_x$. Hence $R\mathcal{C} \cong \oplus_{x \in \mathrm{Ob}\,\mathcal{C}}[R\mathcal{C} \cdot 1_x]^{|[x]|}$, where $|[x]|$ denotes the size of the isomorphism class $[x]$ of each $x \in \mathrm{Ob}\,\mathcal{C}$. These direct summands are projective $R\mathcal{C}$-modules, but in general they are not indecomposable.

Lemma 4.1.1. *Let $\mathcal{C}$ be a finite EI-category. One can write $1_{R\mathcal{C}} = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} \sum_{j=1}^{n_x} e_{xj}$, where n_x is a positive integer for each $x \in \mathrm{Ob}\,\mathcal{C}$ and $1_x = \sum_{j=1}^{n_x} e_{xj}$. As a consequence, $R\mathcal{C} = \oplus_{x \in \mathrm{Ob}\,\mathcal{C}} \oplus_{j=1}^{n_x} R\mathcal{C} \cdot e_{xj}$ for some primitive orthogonal idempotents $e_{xj} \in R\,\mathrm{Aut}_{\mathcal{C}}(x), x \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. We know the identity of $R\mathcal{C}$ is $1 = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} 1_x$, where 1_x is the identity of $R\,\mathrm{Aut}(x)$. Let's fix an object x. Without loss of generality, we suppose $1_x = \sum_{i=1}^{n_x} e_i$, for some primitive orthogonal idempotents e_i of $R\,\mathrm{Aut}(x)$. We show these e_i are primitive idempotents of $R\mathcal{C}$. Suppose some $e_j \in R\,\mathrm{Aut}(x)$ is not primitive. Then it's of the form $a + b$, a sum of two orthogonal idempotents in $R\mathcal{C}$, where both a and b are combinations of morphisms in $\mathcal{C}$. If we write $a = a_1 + a_2 + a_3$ and $b = b_1 + b_2 + b_3$ where $a_1, b_1 \in R\,\mathrm{Aut}(x)$, $a_2, b_2 \in \sum_{y \neq x} R\,\mathrm{Aut}(y)$, and a_3, b_3 are combinations of non-isomorphisms. Using equalities like $a^2 = a$, $b^2 = b$ and $ab = ba = 0$, we can show that a_1 and b_1 are orthogonal idempotents in $R\,\mathrm{Aut}_{\mathcal{C}}(x)$, a_2 and b_2 are orthogonal idempotents in $\sum_{y \neq x} R\,\mathrm{Aut}_{\mathcal{C}}(y)$, while $a_1 + b_1 = e_j$, $a_2 + b_2 = 0$ and $a_3 + b_3 = 0$. So we must have either $a_1 = 0$ or $b_1 = 0$, and $a_2 = b_2 = 0$ by simple calculations. Let's assume $a_1 = 0$. Then $a = a_3$ is an idempotent. From $a_3 + b_3 = 0$ and $a_3^2 = a_3$, we can see $b_3^2 = -b_3$. Now using $ab = 0$ and $ba = 0$ we can deduce $b_3 b_1 = b_3$ and $b_1 b_3 = b_3$. But this is impossible since $b_1 = e_j \in R\,\mathrm{Aut}_{\mathcal{C}}(x)$ while b_3 is a linear combination of non-isomorphisms. Therefore e_j has to be primitive in $R\mathcal{C}$. $\square$

The above lemma gives us some indecomposable projective $R\mathcal{C}$-modules of the form $R\mathcal{C} \cdot e$, where $e \in R\,\mathrm{Aut}_{\mathcal{C}}(x)$ for some $x \in \mathrm{Ob}\,\mathcal{C}$. An indecomposable projective $R\mathcal{C}$-module like that evaluated at x equals $R\,\mathrm{Aut}_{\mathcal{C}}(x) \cdot e$, which is an indecomposable projective $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. If $y \cong x$, then it's not hard to see that $R\mathcal{C} \cdot e(y) = R\,\mathrm{Hom}_{\mathcal{C}}(x, y) \otimes_{R\,\mathrm{Aut}(x)} R\,\mathrm{Aut}(x)e$ is isomorphic to $R\,\mathrm{Aut}(x)e$ as R-modules. These indecomposable projectives are induced modules because for each e the natural surjection $R\mathcal{C} \otimes_{R\,\mathrm{Aut}(x)} R\,\mathrm{Aut}(x) \cdot e \to R\mathcal{C} \cdot e$ has to be an isomorphism.

Corollary 4.1.2. *(Lück [26]) Any projective $R\mathcal{C}$-module is isomorphic to a direct sum of indecomposable projective modules of the form $R\mathcal{C} \cdot e$, where $e \in R\,\mathrm{Aut}_{\mathcal{C}}(x)$ is a primitive idempotent, for some $x \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. Since any projective $R\mathcal{C}$-module is a direct summand of some free $R\mathcal{C}$-module, we can use the preceding lemma and the Krull-Schmidt Theorem to prove the statement. □

Using the above corollary, one can deduce the structures of simple $R\mathcal{C}$-modules, since with the assumption on the base ring, every simple module is a quotient of some indecomposable projective module by its radical. Because each indecomposable projective module is a direct summand of some $R\mathcal{C} \cdot 1_x, x \in \mathrm{Ob}\,\mathcal{C}$, we only have to consider the quotient of $R\mathcal{C} \cdot 1_x$ by its radical in order to obtain a complete list of simple $R\mathcal{C}$-modules. It's not hard to see all the non-isomorphisms in $\mathrm{Hom}_{\mathcal{C}}(x, -)$ span a submodule that is contained in the radical of $R\mathcal{C} \cdot 1_x$. This implies $R\mathcal{C} \cdot 1_x$ is the projective cover of a semi-simple module, which is atomic, concentrated on the isomorphism class $[x]$. With further examination, we can conclude that every simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module gives rise to a simple $R\mathcal{C}$-module, and all simple $R\mathcal{C}$-modules can be obtained in this way.

Theorem 4.1.3. *(Lück [26]) Let $\mathcal{C}$ be an EI-category. For each object $x \in \mathrm{Ob}\,\mathcal{C}$ and simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module V there is a simple $R\mathcal{C}$-module M such that $[x] \in \mathrm{Is}\,\mathcal{C}$ is exactly the set of objects on which M is non-zero, and $M(x) = V$. On the other hand, if M is a simple $R\mathcal{C}$-module, then there exists a unique isomorphism class of objects $[x] \in \mathrm{Is}\,\mathcal{C}$ on which M is non-zero, and furthermore each $M(x)$ is a simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$-module. These two processes are inverse to each other. Thus the isomorphism classes of the simple $R\mathcal{C}$-modules biject with the pairs $([x], V)$, where $x \in \mathrm{Ob}\,\mathcal{C}$ and V is a simple $R\,\mathrm{Aut}_{\mathcal{C}}(x)$ module, taken up to isomorphism.*

Proof. Let M be a simple $R\mathcal{C}$-module and choose $x \in \mathrm{Ob}\,\mathcal{C}$ for which $M(x) \neq 0$. The submodule of M generated by $M(x)$ is non-zero only on objects y with $[x] \leq [y]$, and equals M since M is simple. Therefore $M(y) = 0$ unless $[x] \leq [y]$. The submodule of M generated by the $M(y)$ with $[x] < [y]$ is zero at x, hence is not the whole of M, so must be zero since M is simple. Thus $M(y) = 0$ unless $[x] = [y]$. Finally, if

$M(x)$ were to have a proper submodule as an $R\operatorname{Aut}_{\mathcal{C}}(x)$-module, it would generate a proper submodule of M, so $M(x)$ must be simple $R\operatorname{Aut}_{\mathcal{C}}(x)$-module.

Conversely, given an object $x \in \operatorname{Ob}\mathcal{C}$ and a simple $R\operatorname{Aut}(x)$-module V. Then we can use it to construct an $R\mathcal{C}$-module M as follows.

$$M(y) = \begin{cases} R\operatorname{Hom}(x,y) \otimes_{R\operatorname{Aut}(x)} M(x), & \text{if } y \cong x \\ 0, & \text{otherwise} \end{cases}$$

Thus M has the property that $M(x) = V$ and hence we have shown that there is a bijection between simple $R\mathcal{C}$-modules and pairs $([x], V)$ as claimed. □

We denote a simple $R\mathcal{C}$-module by $S_{x,V}$, if it comes from a simple $R\operatorname{Aut}(x)$-module V, for some $x \in \operatorname{Ob}\mathcal{C}$. For consistency, we use $P_{x,V}$ for the projective cover of $S_{x,V}$, whose structure is determined by its value at the object x. If $R\operatorname{Aut}(x) \cdot e$ is the projective cover of the simple $R\operatorname{Aut}(x)$-module V, then $R\mathcal{C} \cdot e$ is the projective cover of $S_{x,V}$.

Example 4.1.4. *Let $\mathcal{C}$ be a poset and R a field. Then both $[x]$ and $\operatorname{Aut}(x)$ have just one element. All simple modules are of the form $S_{x,1} = R \cdot 1_x$, with projective cover $P_{x,1} = R\mathcal{C} \cdot 1_x$. When we deal with simply a poset, we usually write S_x, P_x for simples and projectives, instead of $S_{x,1}, P_{x,1}$, because there is only one simple $R\mathcal{C}$-module associated to each object of $\mathcal{C}$.*

With the description of indecomposable projectives, we can also show when the trivial module $\underline{R}$ is projective. The following proposition generalizes a result of Symonds [37].

Proposition 4.1.5. *Let $\mathcal{C}$ be a finite EI-category. Then $\underline{R}$ is projective if and only if each connected component of $\mathcal{C}$ has a unique isomorphism class of minimal objects $[x]$, and furthermore such an x has the property that for all y in the same connected component as x, $\operatorname{Aut}(x)$ has a single orbit on $\operatorname{Hom}(x, y)$, and $|\operatorname{Aut}(x)|$ is invertible in R.*

Proof. If $\underline{R}$ is projective then $\underline{R} \cong \bigoplus P_{x,V}$ for certain indecomposable projective modules $P_{x,V}$. The only V which can arise are $V = R$, and R must be projective as an $R\operatorname{Aut}(x)$-module, forcing $|\operatorname{Aut}(x)|$ to be invertible in R for the x which appear in the direct sum, as in the first proof.

Since $P_{y,R}(z) = 0$ unless $y \leq z$, $P_{x,R}$ must appear as a summand for each isomorphism class of minimal x. Now $P_{x,R}(z) = \mathrm{Hom}(x,z) \otimes_{\mathrm{Aut}(x)} R \cong R^n$, where n is the number of orbits of $\mathrm{Aut}(x)$ on $\mathrm{Hom}(x,z)$. For $P_{x,R}(z)$ to be a summand of R we must have $n = 1$.

Finally, $\bigoplus_{\{\text{minimal } x\}} P_{x,R}$ at an object z is R^t where $t =$ number of isomorphism classes of minimal $[x]$ with $x \leq z$, so each component has a unique minimal x.

The other direction is easy. The conditions imply that $\underline{R} = \bigoplus_{\{\text{minimal } x\}} P_{x,R}$ and this is projective. $\square$

Remark 4.1.6. *Since we know the structures of indecomposable projective $R\mathcal{C}$-modules, we can describe the indecomposable injective $R\mathcal{C}$-modules via standard duality, when R is a field. Let P be a projective $R\mathcal{C}$-module. Then $P^* = \mathrm{Hom}_R(P,R)$ is a right injective $R\mathcal{C}$-module. All right injective $R\mathcal{C}$-modules can be obtained in this way, and furthermore if P is indecomposable, so is P^*. We will describe the right $R\mathcal{C}$-module structure of $(R\mathcal{C})^*$ which has every indecomposable injective module as a direct summand, up to isomorphism. It's easy to see $(R\mathcal{C})^*$ is the R-span of the linear functions on $R\mathcal{C}$: $\{f_\alpha | \alpha \in \mathrm{Mor}(\mathcal{C})\}$, where $f_\alpha(\beta) = 0$ for all $\beta \neq \alpha$ and $f_\alpha(\alpha) = 1$. Routine calculations show that the right $R\mathcal{C}$-module structure on $(R\mathcal{C})^*$ is given by*

$$f_\alpha \cdot \beta = \sum\nolimits_{\alpha=\beta\gamma} f_\gamma.$$

The right-hand-side is assumed to be zero if there's no γ satisfying $\alpha = \beta\gamma$. We note that in general the set of right injective $R\mathcal{C}$-modules cannot be identified with the set of right projective $R\mathcal{C}$-modules.

4.2 Restrictions of projective modules

Let M and N be two $R\mathcal{C}$-modules. For computing $\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$, we plan to develop some machinery which reduces the size of $\mathcal{C}$, by which we mean finding a (full) subcategory $\mathcal{D} \subset \mathcal{C}$ such that $\mathrm{Ext}^*_{R\mathcal{C}}(M,N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(\mathrm{Res}_\iota M, \mathrm{Res}_\iota N)$. This isomorphism will be established using the Eckmann-Shapiro Lemma and the theory of vertices and sources which we shall develop in the next two sections. In order to apply the Eckmann-Shapiro Lemma we have to require, for the given inclusion $\iota : \mathcal{D} \hookrightarrow C$, the functor $\uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} -$ to preserve the projective resolutions of $R\mathcal{D}$-modules. The functor $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ always sends projective modules to projective modules so we just need to find out when it's exact.

If $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module then $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ is exact. In our context, a feasible way to sort out the flatness of $R\mathcal{C}$ is to figure out when it is a right projective $R\mathcal{D}$-module. However in order to be consistent with our general set-up, we will describe how to make $R\mathcal{C}$ into a left projective $R\mathcal{D}$-module. We note that the argument for $R\mathcal{C}$ being a right projective $R\mathcal{D}$-module follows immediately in a dual fashion. This means after present section we will know for which full subcategories $\downarrow_{\mathcal{D}}^{\mathcal{C}}$ preserves left or right projective $R\mathcal{C}$-modules.

We find it's convenient to study the action of $\downarrow_{\mathcal{D}}^{\mathcal{C}} : (R\text{-mod})^{\mathcal{C}} \to (R\text{-mod})^{\mathcal{D}}$ on the representable functors in order to formulate the conditions on $\mathcal{D}$ for which $\downarrow_{\mathcal{D}}^{\mathcal{C}}$ preserves (left and right) projective modules. We will concent ourselves with the full subcategories of $\mathcal{C}$, for some reasons we will see soon in the course of developing our theory of vertices and sources.

The main theorems here are due to tom Dieck [9]. In order to use the characterization of projective modules, we assume R is a field or a complete discrete valuation ring.

Definition 4.2.1. *Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. Then we say a non-isomorphism $f \in \mathrm{Mor}(\mathcal{C})$ is irreducible (resp. co-irreducible) with respect to $\mathcal{D}$ if there's no way to write $f = f_1 \cdot f_2$ with f_1, f_2 non-isomorphisms and f_1 (resp. f_2) $\in \mathrm{Mor}(\mathcal{D})$.*

Given $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ and $y \in \mathrm{Ob}\,\mathcal{D}$ the subset of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of all non-isomorphisms which are irreducible with respect to $\mathcal{D}$ is denoted by $\mathrm{Irr}_{\mathcal{D}}(x, y)$. Similarly given $x \in \mathrm{Ob}\,\mathcal{D}$ and $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ we define $\mathrm{coIrr}_{\mathcal{D}}(x, y)$ to be the subset of $\mathrm{Hom}_{\mathcal{C}}(x, y)$ consisting of non-isomorphisms that are co-irreducible with respect to $\mathcal{D}$.

If a non-isomorphism $f \in \mathrm{Mor}(\mathcal{C})$ is irreducible with respect to $\mathcal{C}$, then we simply say f is irreducible. Note that our irreducible morphisms are different from those in the representation theory of Artin algebras, see [2].

Lemma 4.2.2. *Let $\mathcal{D}$ be a full subcategory of an EI-category $\mathcal{C}$. Suppose $x \in \mathrm{Ob}\,\mathcal{C}, y \in \mathrm{Ob}\,\mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is not empty. Then as an $R\,\mathrm{Aut}(y)$-module,*

$$R\mathrm{Hom}_{\mathcal{C}}(x, y) = R\mathrm{Irr}_{\mathcal{D}}(x, y) \oplus R\,\mathrm{r}(x, y),$$

where $\mathrm{Irr}_{\mathcal{D}}(x,y)$ *is the set of all non-isomorphisms which are irreducible with respect to* $\mathcal{D}$*, and* $\mathrm{r}(x,y)$ *is the complementary set of* $\mathrm{Irr}_{\mathcal{D}}(x,y) \subset \mathrm{Hom}_{\mathcal{C}}(x,y)$*. Of course either of them could be empty.*

Proof. Obvious. □

Recall that we have made the following assumption: if $\mathcal{D} \subset \mathcal{C}$ is a full subcategory then it satisfies the condition that if $y \in \mathrm{Ob}\,\mathcal{D}$ then $[y] \subset \mathrm{Ob}\,\mathcal{D}$, where $[y]$ is the isomorphism class of $y \in \mathrm{Ob}\,\mathcal{C}$. Let $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)$ be a representable functor for some $x \in \mathrm{Ob}\,\mathcal{C}$. We consider the restriction of it: $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$. When $[x] \subset \mathrm{Ob}\,\mathcal{D}$, we simply have $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\,\mathrm{Hom}_{\mathcal{D}}(x,-)$, which is a projective $R\mathcal{D}$-module. When $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$, we have

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \sum\nolimits_{y\in\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y), \qquad (*)$$

where Ψ_x is the set of all $y \in \mathrm{Ob}\,\mathcal{D}_{\geq x} = \mathrm{Ob}\,\mathcal{D}_{>x}$ such that $\mathrm{Irr}_{\mathcal{D}}(x,y)$ is non-empty. It is so because, for each $z \in \mathrm{Ob}\,\mathcal{D}_{\geq x}$, every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x,z)$ has a suitable factorization $f = f_1 \cdot f_2$ for some $f_2 \in \mathrm{Irr}_{\mathcal{D}}(x,y), y \in \mathrm{Ob}\,\mathcal{D}_{\geq x}$ (y doesn't have to be minimal in $\mathrm{Ob}\,\mathcal{D}_{\geq x}$!). One can see that if $y, y' \in \Psi_x$ and $y \cong y'$, then $R\,\mathrm{Irr}_{\mathcal{D}}(x,y) \cong R\,\mathrm{Irr}_{\mathcal{D}}(x,y')$ and $R\,\mathrm{Hom}(y,-)R\,\mathrm{Irr}_{\mathcal{D}}(x,y) = R\,\mathrm{Hom}(y',-)R\,\mathrm{Irr}_{\mathcal{D}}(x,y')$, and thus

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \sum\nolimits_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y). \qquad (*)$$

Here we regard $R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)$ as a covariant functor on $\mathcal{D}$ such that

$$[R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)](z) = R\mathrm{Hom}_{\mathcal{D}}(y,z)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y).$$

We will keep using this kind of expressions even though it looks like a "product" of a functor and a module. If $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is a projective $R\mathcal{D}$-module, then we show the right-hand-side of the above equality $(*)$ must be a direct sum.

Lemma 4.2.3. *Let* $\mathcal{D}$ *be a full subcategory of a finite EI-category* $\mathcal{C}$*. Suppose* $x \in \mathrm{Ob}\,\mathcal{C}$ *and* $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$*. If* $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ *is projective, then we must have*

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus\nolimits_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y),$$

and thus $R\,\mathrm{Hom}_{\mathcal{D}}(y,-)R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ *is a projective* $R\mathcal{D}$*-module for every* $y \in \Psi_x$*.*

Proof. We can write $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus_{z,V} M_{z,V}$ for some $z \in \mathrm{Ob}\,\mathcal{D}$ and simple $R\,\mathrm{Aut}(z)$-modules V, with $M_{z,V} \cong P_{z,V}$. Given $w \in \mathrm{Ob}\,\mathcal{D}$, $M_{z,V}(w)$ consists of elements which are linear combinations of morphisms in $\mathrm{Hom}_{\mathcal{C}}(x,w)$. Note that from the structure of indecomposable projective modules (Corollary 4.1.2), every $M_{z,V}$ is determined by $M_{z,V}(z)$ (an indecomposable projective $R\,\mathrm{Aut}(z)$-module), and that $M_{z,V}(z') = 0$ if $z' \not\geq z$. It implies that $M_{z,V}(w) \subset R\,\mathrm{Hom}_{\mathcal{C}}(x,w)$ consists of linear combinations of morphisms which factor through z.

First of all, we show that whenever $\mathrm{Irr}_{\mathcal{D}}(x,y) \neq \emptyset$ there exists some $M_{y,V}$ as a direct summand of $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$. The thing is

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)(y)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus_{z,V} M_{z,V}(y) = \{\bigoplus_{z\cong y} M_{z,V}(y)\}\bigoplus\{\bigoplus_{z<y} M_{z,V}(y)\}.$$

Since the second summand is a submodule of $R\,\mathrm{r}(x,y)$, we must have $M_{z,V}(y) \neq 0$ for some $z \cong y$.

Second, we'll prove, for every $y \in \mathrm{Ob}\,\mathcal{D}_{>x}$, $\bigoplus_{z\cong y} M_{z,V}(y) \cong R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$. Suppose $w \in \mathrm{Ob}\,\mathcal{D}_{>x}$ is a minimal object. Then it's easy to see $R\,\mathrm{Irr}_{\mathcal{D}}(x,w) = R\,\mathrm{Hom}_{\mathcal{C}}(x,w) \neq 0$ and $\bigoplus_{z\cong w} M_{z,V}(w) = R\,\mathrm{Hom}_{\mathcal{C}}(x,w) = R\,\mathrm{Irr}_{\mathcal{D}}(x,w)$. Let $\mathcal{M}_0$ be the set of minimal objects in $\mathrm{Ob}\,\mathcal{D}_{>x}$. We consider the minimal objects of $\mathrm{Ob}\,\mathcal{D}_{>x}\backslash\mathcal{M}_0$. For every $u \in \mathrm{Ob}\,\mathcal{D}_{>x}\backslash\mathcal{M}_0$, we'll show $\bigoplus_{z\cong u} M_{z,V}(u) \cong R\,\mathrm{Irr}_{\mathcal{D}}(x,u)$ too. Then inductively we can prove, for every $y \in \mathrm{Ob}\,\mathcal{D}$, $\bigoplus_{z\cong y} M_{z,V}(y) \cong R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$, since $\mathrm{Ob}\,\mathcal{D}$ is finite. As usual we consider the equality $R\,\mathrm{Hom}_{\mathcal{C}}(x,u) = \{\bigoplus_{z\cong u} M_{z,V}(u)\}\bigoplus\{\bigoplus_{z<u} M_{z,V}(u)\}$. We have the second summand $\bigoplus_{z<u} M_{z,V}(u) = \bigoplus_{z\in\mathcal{M}_0} M_{z,V}$ equals $R\,\mathrm{r}(x,u)$, and therefore from

$$R\mathrm{Irr}_{\mathcal{D}}(x,u)\bigoplus R\,\mathrm{r}(x,u) = \{\bigoplus_{z\cong u} M_{z,V}(u)\}\bigoplus\{\bigoplus_{z\in\mathcal{M}_0} M_{z,V}(u)\},$$

we obtain $R\,\mathrm{Irr}_{\mathcal{D}}(x,u) \cong \bigoplus_{z\cong u} M_{z,V}(u)$.

In the end from $\bigoplus_{z\cong y} M_{z,V}(y) \cong R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ we get

$$R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot\bigoplus_{z\cong y} M_{z,V}(y) = \bigoplus_{z\cong y} M_{z,V} \cong R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y).$$

Hence we have an isomorphism $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} \cong \bigoplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)$. Along with the equality $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \sum_{[y]\subset\Psi_x} R\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$, it implies

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y),$$

because the R-ranks of $\sum_{[y]\subset\Psi_x} R\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ and $\oplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)$ are equal. $\square$

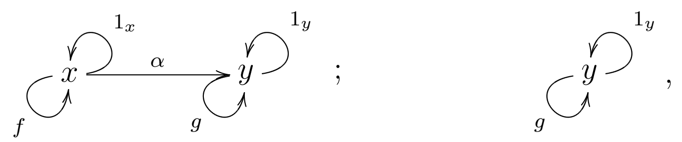
Remark 4.2.4. *In fact we can continue to show every $R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ in Lemma 4.2.3 is a projective $R\,\mathrm{Aut}(y)$-module (this is recorded as Lemma 4.2.8). Meanwhile, it's not hard to see that*

$$\bigoplus_{z\cong y,V} M_{z,V} = \bigoplus_{z\cong y}\{\bigoplus_{V} M_{z,V}\} \cong |[y]|\cdot\bigoplus_{V} M_{y,V}.$$

Here $|[y]|$ denotes the size of the isomorphism class $[y]$. The reason is that $M_{y,V}$ is determined by $M_{y,V}(y)$ and every $g\in\mathrm{Is}(y,z)$ induces a permutation of direct summands of the leftmost term evaluated at y, because $gR\,\mathrm{Irr}_{\mathcal{D}}(x,y) = R\,\mathrm{Irr}_{\mathcal{D}}(x,z) = R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$. More explicitly, the action of g sends a direct summand, on evaluation at y, $\bigoplus_V M_{y,V}(y) = \bigoplus_V R\,\mathrm{Aut}(y)\cdot b_{y,V}$ isomorphically to $\bigoplus_V gR\,\mathrm{Aut}(y)g^{-1}\cdot gb_{y,V} = \bigoplus_V R\,\mathrm{Aut}(z)\cdot(gb_{y,V})$, which equals $\bigoplus_{V'} M_{z,V'}(z)$. Here $b_{y,V}\in R\,\mathrm{Hom}_{\mathcal{C}}(x,y)$ is an element which satisfies $M_{y,V}(y) = R\,\mathrm{Aut}(y)\cdot b_{y,V}$ for each V.

Before we go further, we present two simple examples where $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projectives.

Example 4.2.5. *First, we consider the following category $\mathcal{C}$ and its full subcategory $\mathcal{D}$*



where $g\alpha=\alpha, \alpha f=\alpha, f^2=1_x$ and $g^2=1_y$. Suppose $R=\mathbb{C}$. We can get $\mathbb{C}\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \mathbb{C}\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot\mathbb{C}\,\mathrm{Irr}_{\mathcal{D}}(x,y) = \mathbb{C}\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot\alpha \cong \mathbb{C}\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot\frac{(1_y+g)}{2}$ and $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\,\mathbb{C}\,\mathrm{Hom}_{\mathcal{C}}(y,-) = \mathbb{C}\,\mathrm{Hom}_{\mathcal{D}}(y,-)$, which are both projective $\mathbb{C}\mathcal{D}$-modules. (Note if R is a field of characteristic 2, then $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is not projective.)

Second, we consider the category $\mathcal{C}'$ and its subcategory $\mathcal{D}'$

where $\gamma\alpha \neq \mu\beta$. We have $R\,\mathrm{Hom}(w,-)\downarrow^{\mathcal{C}'}_{\mathcal{D}'} \cong R\,\mathrm{Hom}_{\mathcal{D}'}(x,-) \oplus R\,\mathrm{Hom}_{\mathcal{D}'}(y,-)$ due to the fact $\gamma\alpha \neq \mu\beta$. We need to emphasize that without the assumption $\gamma\alpha \neq \mu\beta$ the isomorphism does not exist.

From the first example we know $R\,\mathrm{Hom}_{\mathcal{C}}(x,y) = R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ is projective as an $R\,\mathrm{Aut}(y)$-module, which verifies the assertion of the first sentence in Remark 4.2.4. We shall show $R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ being projective, when x isn't isomorphic to an object of $\mathcal{D}$, is necessary for $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ to preserve projectives. From the second example we see a certain factorization property of the non-isomorphisms is required for $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ to preserve projectives. The factorization property actually guarantees the sum $(*)$ which appears before Lemma 4.2.3 to be direct. Thus the examples tell us both the base ring and the intrinsic structure of $\mathcal{C}$ will affect the action of $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ on representable functors.

Definition 4.2.6. *Let $x \notin \mathrm{Ob}\,\mathcal{D}$ and $z \in \mathrm{Ob}\,\mathcal{D}$. Suppose $f \in \mathrm{Hom}_{\mathcal{C}}(x,z)$ and f admits two factorizations $f = f_1 \cdot f_2 = f_1' \cdot f_2'$ with $f_1, f_1' \in \mathrm{Mor}(\mathcal{D})$ and f_2, f_2' irreducible with respect to $\mathcal{D}$. Then we say f is good if for every such pair of factorizations f_2, f_2' have isomorphic co-domains. We say $\mathcal{D}$ is good with respect to $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ if for all $z \in \mathrm{Ob}\,\mathcal{D}$ with $\mathrm{Hom}_{\mathcal{C}}(x,z) \neq \emptyset$, all $f \in \mathrm{Hom}_{\mathcal{C}}(x,z)$ are good. We say $\mathcal{D}$ is good if $\mathcal{D}$ is good with respect to all $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$.*

Lemma 4.2.7. *Let $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$. The full subcategory $\mathcal{D}$ is good with respect to x if and only if*

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus\nolimits_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y),$$

where Ψ_x is the union of all isomorphism classes $[y] \subset \mathrm{Ob}\,\mathcal{D}_{\geq x}$ such that $\mathrm{Irr}_{\mathcal{D}}(x,y)$ is non-empty.

Proof. Given the goodness we show the intersection

$$(R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)) \bigcap (\sum_{y' \ncong y} R\mathrm{Hom}_{\mathcal{D}}(y',-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y')) = \{0\}.$$

For proving this, it suffices to show that the intersection is zero on evaluation at each $z \in \mathrm{Ob}\,\mathcal{D}$. Suppose

$$u \in [(R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)) \bigcap (\sum_{y' \neq y} R\mathrm{Hom}_{\mathcal{D}}(y',-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y'))](z).$$

Then we can write $u = \sum_i r_i u_i^1 \cdot u_i^2$ with $u_i^1 \in \mathrm{Hom}_{\mathcal{D}}(y,z)$ and $u_i^2 \in \mathrm{Irr}_{\mathcal{D}}(x,y)$ for some $y \in \mathrm{Ob}\,\mathcal{D}_{\geq x}$. Since $u \in \sum_{y' \ncong y} R\,\mathrm{Hom}_{\mathcal{D}}(y',-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y')$, we can also put $u = \sum r'_j u'^1_j \cdot u'^2_j$, with $u'^1_j \in \mathrm{Hom}_{\mathcal{D}}(y',z)$ and $u'^2_j \in \mathrm{Irr}_{\mathcal{D}}(x,y')$ for $y' \ncong y$ and $y' \in \mathrm{Ob}\,\mathcal{D}_{\geq x}$. Therefore

$$\sum_i r_i u_i^1 \cdot u_i^2 = \sum_j r'_j u'^1_j \cdot u'^2_j.$$

But the equality implies that r_i, r'_j are all zero for our condition guarantees $u_i^1 \cdot u_i^2 \neq u'^1_j \cdot u'^2_j$ for any i, j since u_i^2, u_j^2 have non-isomorphic targets, and morphisms are independent in $R\,\mathrm{Hom}_{\mathcal{C}}(x,z)$, by definition. Hence $u = 0$, and we have the direct sum.

On the other hand, if

$$R\mathrm{Hom}_{\mathcal{C}}(x,-) \downarrow_{\mathcal{D}}^{\mathcal{C}} = \bigoplus_{[y] \subset \Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y),$$

and $f = f_1 \cdot f_2 = f'_1 \cdot f'_2 \in \mathrm{Hom}_{\mathcal{C}}(x,z)$ with $f_1 \in \mathrm{Hom}_{\mathcal{D}}(y,z), f'_1 \in \mathrm{Hom}_{\mathcal{D}}(y',z), f_2 \in \mathrm{Irr}_{\mathcal{D}}(x,y)$ and $f'_2 \in \mathrm{Irr}_{\mathcal{D}}(x,y')$, then we show $y \cong y'$. For otherwise from $f_2 \in R\,\mathrm{Hom}(y,-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ and $f'_2 \in R\,\mathrm{Hom}(y',-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y')$, we get

$$f \in R\,\mathrm{Hom}(y,z) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y) \cap R\,\mathrm{Hom}(y',z) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y'),$$

which is a contradiction. □

Next we examine when the $R\mathcal{D}$-module $R\,\mathrm{Hom}(y,-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ is projective.

Lemma 4.2.8. *Let $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$. Then the $R\mathcal{D}$-module $R\,\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ being projective implies $R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ is projective as an $R\,\mathrm{Aut}(y)$-module.*

Proof. If $R\,\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ is a projective $R\mathcal{D}$-module, then from the structure theorem of the projective modules

$$R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\mathrm{Irr}_{\mathcal{D}}(x,y) \cong \bigoplus_i R\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot e_i$$

for some primitive idempotents e_i of $R\,\mathrm{Aut}(y)$. Evaluating the two functors on both sides at y, we get $[R\,\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x,y)](y) \cong [\bigoplus_i R\,\mathrm{Hom}_{\mathcal{D}}(y,-) \cdot e_i](y)$, or $R\,\mathrm{Irr}_{\mathcal{D}}(x,y) \cong \bigoplus_i R\,\mathrm{Aut}(y) e_i$. □

The converse of the above lemma is not true.

Example 4.2.9. *Let $\mathcal{C}$ be the following category*

$$x \underset{\beta}{\overset{\alpha}{\rightrightarrows}} y \xrightarrow{\ \gamma\ } z\ ,$$

with $\gamma\alpha = \gamma\beta$. Consider the full subcategory $\mathcal{D}$

$$y \xrightarrow{\ \gamma\ } z\ .$$

We have $R\operatorname{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\operatorname{Hom}_{\mathcal{D}}(y,-)\cdot R\operatorname{Irr}_{\mathcal{D}}(x,y)$ with $\operatorname{Hom}(y,-) = \{1_y,\gamma\}$ and $\operatorname{Irr}_{\mathcal{D}}(x,y) = \{\alpha,\beta\}$. Obviously, $R\operatorname{Irr}_{\mathcal{D}}(x,y) = R\alpha \oplus R\beta$ is a free $R\operatorname{Aut}(y)$-module. But $R\operatorname{Hom}(y,-)\cdot R\operatorname{Irr}_{\mathcal{D}}(x,y)$ is not a projective $R\mathcal{D}$-module since it equals $R\operatorname{Hom}_{\mathcal{D}}(y,-)\{\frac{\alpha+\beta}{2}\} \oplus R\operatorname{Hom}_{\mathcal{D}}(y,-)\{\frac{\alpha-\beta}{2}\}$ (suppose $charR \neq 2$), while the first summand $R\operatorname{Hom}_{\mathcal{D}}(y,-)\{\frac{\alpha+\beta}{2}\} \cong R\operatorname{Hom}_{\mathcal{D}}(y,-)$ is projective and second summand $R\operatorname{Hom}_{\mathcal{D}}(y,-)\{\frac{\alpha-\beta}{2}\} \cong S_y$ is not.

Suppose $R\operatorname{Irr}_{\mathcal{D}}(x,y) \cong \bigoplus_i R\operatorname{Aut}(y)e_i$. In order to maintain a direct sum on the right-hand-side after multiplying both sides with $R\operatorname{Hom}_{\mathcal{D}}(y,-)$, we need an extra condition, which we borrow from tom Dieck [9].

Lemma 4.2.10. *Suppose $[x]\cap \operatorname{Ob}\mathcal{D} = \emptyset$ and $R\operatorname{Irr}_{\mathcal{D}}(x,y)$ is a projective $R\operatorname{Aut}(y)$-module. Then the $R\mathcal{D}$-module $R\operatorname{Hom}_{\mathcal{D}}(y,-)\cdot R\operatorname{Irr}_{\mathcal{D}}(x,y)$ is projective if and only if*

$$R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y) \cong R\mathrm{Hom}_{\mathcal{D}}(y,-)\otimes_{R\operatorname{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x,y).$$

Proof. The "if" part is easy since the tensor product $R\operatorname{Hom}_{\mathcal{D}}(y,-)\otimes_{R\operatorname{Aut}(y)} -$ preserves projectives. The "only if" part can be obtained by considering the natural transformation

$$\pi : R\mathrm{Hom}_{\mathcal{D}}(y,-)\otimes_{R\operatorname{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x,y) \to R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y),$$

given on each object $z \in \operatorname{Ob}\mathcal{D}$ via multiplication

$$\pi_z : R\mathrm{Hom}_{\mathcal{D}}(y,z)\otimes_{R\operatorname{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x,y) \to R\mathrm{Hom}_{\mathcal{D}}(y,z)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y).$$

Every π_z is a splitting epimorphism because both ends are projective modules. Furthermore π_y is an isomorphism. Since both functors are determined by their evaluations at y, the natural transformation π is indeed an isomorphism. □

We also have a combinatorial condition on the existence of the isomorphism in the previous lemma (compare with Lemma 4.2.6). It was formulated by tom Dieck ([9], 11.38(iii)), and it helps us to improve our written statement of the main result in this section. The condition (**TD**) is stronger than our definition of goodness.

(**TD**) Let $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}, y \in \mathrm{Ob}\,\mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. If f admits two factorizations $f_1 \cdot f_2 = f_1' \cdot f_2'$ with f_2, f_2' irreducible with respect to $\mathcal{D}$, then there is an isomorphism $g \in \mathrm{Mor}(\mathcal{D})$ such that $f_2 = g f_2'$ and $g f_1 = f_1'$.

The following lemma is actually extracted from tom Dieck's Proposition 11.39 [9].

Lemma 4.2.11. *Let $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}, y \in \mathrm{Ob}\,\mathcal{D}$. Then*

$$R\mathrm{Hom}_{\mathcal{D}}(y, -) \otimes_{R\,\mathrm{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x, y) \cong R\mathrm{Hom}_{\mathcal{D}}(y, -) \cdot R\mathrm{Irr}_{\mathcal{D}}(x, y),$$

*if and only if every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(y, z)\,\mathrm{Irr}_{\mathcal{D}}(x, y), z \in \mathrm{Ob}\,\mathcal{D}$ satisfies condition (**TD**).*

Proof. Again we consider the natural transformation $\pi = \{\pi_z\}_{z \in \mathrm{Ob}\,\mathcal{D}}$. It's always an epimorphism, and it's an isomorphism if and only if (**TD**) is satisfied. □

The previous two lemmas imply that, given the condition (**TD**), $R\,\mathrm{Hom}(y, -) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x, y)$ is projective if $R\,\mathrm{Irr}_{\mathcal{D}}(x, y)$ is. We write this out as a corollary.

Corollary 4.2.12. *Suppose $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ and $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ with $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$. If all morphisms in $\mathrm{Hom}_{\mathcal{C}}(x, z), z \in \mathrm{Ob}\,\mathcal{D}$ satisfy condition (**TD**), then $R\,\mathrm{Hom}(y, -) \cdot R\,\mathrm{Irr}_{\mathcal{D}}(x, y)$ is projective if and only if $R\,\mathrm{Irr}_{\mathcal{D}}(x, y)$ is projective.*

Proof. Just combine Lemmas 4.2.8, 4.2.10 and 4.2.11. □

At this point, we can see, just assuming condition (**TD**), we have

$$\begin{aligned} R\mathrm{Hom}_{\mathcal{C}}(x, -) \downarrow^{\mathcal{C}}_{\mathcal{D}} &= \bigoplus\nolimits_{[y] \subset \Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y, -) \cdot R\mathrm{Irr}_{\mathcal{D}}(x, y) \\ &\cong \bigoplus\nolimits_{[y] \subset \Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y, -) \otimes_{R\,\mathrm{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x, y). \end{aligned}$$

To summarize the results, we state the following theorem.

Theorem 4.2.13. *Suppose R is a field or a complete discrete valuation ring. Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory and $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$. Then $R\,\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is projective if and only if the following conditions are satisfied:*

1. *for each $y \in \mathrm{Ob}\,\mathcal{D}$, every $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ satisfies the condition* **(TD)**; *and*

2. *for each $y \in \mathrm{Ob}\,\mathcal{D}$, $R\,\mathrm{Irr}_{\mathcal{D}}(x,y)$ is a projective $R\,\mathrm{Aut}(y)$-module.*

Proof. When $R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is projective, we have

$$R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \bigoplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)$$

by Lemma 4.2.3. Hence both $R\,\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y)$ and $R\mathrm{Irr}_{\mathcal{D}}(x,y)$ are projective by Lemmas 4.2.4 and 4.2.8, respectively. In the end from Lemmas 4.2.10 and 4.2.11, we know all morphisms satisfy the condition (**TD**).

On the other hand, with condition (**TD**) we have

$$\begin{aligned} R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} &= \textstyle\bigoplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\cdot R\mathrm{Irr}_{\mathcal{D}}(x,y) \\ &\cong \textstyle\bigoplus_{[y]\subset\Psi_x} R\mathrm{Hom}_{\mathcal{D}}(y,-)\otimes_{R\,\mathrm{Aut}(y)} R\mathrm{Irr}_{\mathcal{D}}(x,y). \end{aligned}$$

This along with condition (2) gives the projectivity of $R\mathrm{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$. □

We borrow the notion of "tautness" (with slight modifications) from Symonds [37].

Definition 4.2.14. *We say a full subcategory $\mathcal{D}$ is R-taut in $\mathcal{C}$ if for every $x \in \mathrm{Ob}\,\mathcal{C}$ with $[x] \cap \mathcal{D} = \emptyset$, $\mathcal{D}$ meets the conditions (1) and (2) in Theorem 4.2.13.*

When $\mathcal{D} \subset \mathcal{C}$ is R-taut, $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projectives by Theorem 4.2.13 because any projective $R\mathcal{C}$-module is isomorphic to a direct summand of some $R\mathcal{C}^n$ and $R\mathcal{C}$ is a direct sum of representable functors. Next we mention a sufficient condition for $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ to preserve projectives.

Proposition 4.2.15. *Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory satisfying the following conditions:*

1. *For any two morphisms $f_1, f_2 \in \mathrm{Irr}_{\mathcal{D}}(x,y)$, if $g_1 f_1 = g_2 f_2$ for some $g_1, g_2 \in \mathrm{Mor}(\mathcal{D})$ then $f_1 = g f_2, g g_1 = g_2$ for some $g \in \mathrm{Is}(\mathcal{D})$; and*

2. $|Stab_{\mathrm{Aut}(y)}(f)|^{-1} \in R$ *for any* $f \in \mathrm{Irr}_{\mathcal{D}}(x, y)$.

Then $\mathcal{D} \subset \mathcal{C}$ *is R-taut.*

Proof. In fact (1) implies (**TD**), and (2) says that $R\,\mathrm{Irr}_{\mathcal{D}}(x, y)$ is a projective $R\,\mathrm{Aut}(y)$-module due to a standard result on permutation modules of finite groups, and that the condition (2) of the previous theorem is satisfied. □

For future reference, we give a dual version of Theorem 4.2.13. It's easily to see that the *right* indecomposable projective $R\mathcal{C}$-modules are of the form $e \cdot R\,\mathrm{Hom}_{\mathcal{C}}(-, x)$ for a primitive idempotent $e \in R\,\mathrm{Aut}(x)$, $x \in \mathrm{Ob}\,\mathcal{C}$. Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory satisfying the condition that if $y \in \mathrm{Ob}\,\mathcal{D}$ then $[y] \subset \mathrm{Ob}\,\mathcal{D}$. As before, we consider the restriction on the representable functor $R\,\mathrm{Hom}_{\mathcal{C}}(-, x)$. If $x \in \mathrm{Ob}\,\mathcal{D}$ then we simply have $R\,\mathrm{Hom}_{\mathcal{C}}(-, x)\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\,\mathrm{Hom}_{\mathcal{D}}(-, x)$. When $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ we have

$$R\mathrm{Hom}_{\mathcal{C}}(-, x)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \sum_{[y] \subset \Psi'_x} R\mathrm{coIrr}_{\mathcal{D}}(y, x) \cdot R\mathrm{Hom}_{\mathcal{D}}(-, y),$$

where Ψ'_x is the set of all $y \in \mathrm{Ob}\,\mathcal{D}$ such that $\mathrm{coIrr}_{\mathcal{D}}(y, x) \neq \emptyset$.

In order to describe $\downarrow^{\mathcal{C}}_{\mathcal{D}}$, we need introduce the notion of co-goodness. Let $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$. Suppose $f \in \mathrm{Hom}_{\mathcal{C}}(z, x), z \in \mathrm{Ob}\,\mathcal{D}$ and f admits two different factorizations $f = f_1 \cdot f_2 = f'_1 \cdot f'_2$ with $f_2, f'_2 \in \mathrm{Mor}(\mathcal{D})$ and f_1, f'_1 *irreducible* with respect to $\mathcal{D}$. Then we say f is *co-good* if f_1, f'_1 have isomorphic domains. We say $\mathcal{D}$ is *co-good with respect to* $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ if for all $z \in \mathrm{Ob}\,\mathcal{D}$, all $f \in \mathrm{Hom}_{\mathcal{C}}(z, x)$ are good, provided $\mathrm{Hom}_{\mathcal{C}}(z, x) \neq \emptyset$. We say $\mathcal{D}$ is *co-good* if $\mathcal{D}$ is co-good with respect to all $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$. There is also a dual condition to (**TD**), and for consistency we call it (**co-TD**), while its formulation is left for the readers.

Theorem 4.2.16. *Suppose R is a field or a complete discrete valuation ring. Let* $\mathcal{D} \subset \mathcal{C}$ *be a full subcategory and* $x \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$. *Then* $R\,\mathrm{Hom}_{\mathcal{C}}(-, x)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ *is a right projective* $R\mathcal{D}$*-module if and only if the following conditions are satisfied:*

1. for each $y \in \mathrm{Ob}\,\mathcal{D}$, *every* $f \in \mathrm{Hom}_{\mathcal{C}}(y, x)$ *satisfies the condition (***co-TD***); and*

2. for each $y \in \mathrm{Ob}\,\mathcal{D}$, $R\,\mathrm{coIrr}_{\mathcal{D}}(y, x)$ *is a right projective* $R\,\mathrm{Aut}(y)$*-module.*

Definition 4.2.17. *We say* $\mathcal{D}$ *is R-co-taut in* $\mathcal{C}$ *if for every* $x \in \mathrm{Ob}\,\mathcal{C}$ *with* $[x] \cap \mathrm{Ob}\,\mathcal{D} = \emptyset$, $\mathcal{D}$ *meets the conditions (1) and (2) in Theorem 4.2.16.*

We show the existence of such R-taut and R-co-taut subcategories by constructing two classes of subcategories which we will work with later on.

Definition 4.2.18. *Suppose $\mathcal{D} \subset \mathcal{C}$ is a (full) subcategory. We say $\mathcal{D}$ is an ideal in $\mathcal{C}$ if, for any $x \in \operatorname{Ob}\mathcal{D}$, $\mathcal{C}_{\leq x} \subset \mathcal{D}$. Similarly, we say $\mathcal{D}$ is a co-ideal in $\mathcal{C}$ if, for any $x \in \operatorname{Ob}\mathcal{D}$, we have $\mathcal{C}_{\geq x} \subset \mathcal{D}$.*

Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory. Then we can form a full subcategory of $\mathcal{C}$, named $\mathcal{C}\backslash\mathcal{D}$, which consists of all objects not belonging to $\mathcal{D}$. From the definitions it's easy to verify that if $\mathcal{D}$ is an ideal (resp. a co-ideal) then $\mathcal{C}\backslash\mathcal{D}$ is a co-ideal (resp. an ideal). Note that if $\mathcal{D} \subset \mathcal{C}$ is an ideal (resp. a co-ideal) then $R\mathcal{D}$ becomes a right ideal (resp. a left ideal) in $R\mathcal{C}$.

If a full subcategory $\mathcal{D}$ forms an ideal (resp. a co-ideal) in $\mathcal{C}$, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves projectives (resp. right projectives). These can be regarded as trivial cases of Theorems 4.2.13 and 4.2.16.

Lemma 4.2.19. *If $\mathcal{D}$ is an ideal in $\operatorname{Ob}\mathcal{C}$, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves left projective modules. If $\mathcal{D}$ is a co-ideal in $\operatorname{Ob}\mathcal{C}$, then $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves right projective modules.*

Proof. We only prove the first assertion by computing $R\operatorname{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}}$ explicitly. If $x \in \operatorname{Ob}\mathcal{D}$, then $R\operatorname{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = R\operatorname{Hom}_{\mathcal{D}}(x,-)$. If $x \notin \operatorname{Ob}\mathcal{D}$, then by definition $R\operatorname{Hom}_{\mathcal{C}}(x,-)\downarrow^{\mathcal{C}}_{\mathcal{D}} = 0$. Hence $R\operatorname{Hom}_{\mathcal{C}}(x,-)$, and consequently $R\mathcal{C}$, are projective $R\mathcal{D}$-modules. □

We record the essence of the argument used to prove Lemma 4.2.19 for future reference.

Corollary 4.2.20. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ which is an ideal (resp. a co-ideal) in $\operatorname{Ob}\mathcal{C}$. We have $R\mathcal{C}\downarrow^{\mathcal{C}}_{\mathcal{D}} \cong R\mathcal{D}$ as an $R\mathcal{D}$-module (resp. a right $R\mathcal{D}$-module).*

Remark 4.2.21. *As for the injective modules, from*

$$\operatorname{Hom}_R(R\mathcal{C}, R)\downarrow^{\mathcal{C}}_{\mathcal{D}} = \operatorname{Hom}_R(R\mathcal{C}\downarrow^{\mathcal{C}}_{\mathcal{D}}, R)$$

as right $R\mathcal{D}$-modules, we can see that $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves right injective $R\mathcal{C}$-modules if and only if $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves left projective $R\mathcal{C}$-modules, i.e., $\mathcal{D}$ is R-taut. (Here we assume R is a field.) In fact, when $\mathcal{D}$ is an ideal in $\mathcal{C}$, we can see this by direct computation.

The thing is, the right $R\mathcal{D}$-module $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}(R\mathcal{C})^*$ decomposes into a direct sum as follows (see Remark 4.1.6 for the structure of $(R\mathcal{C})^*$)

$$\begin{aligned}(R\mathcal{C})^* \downarrow^{\mathcal{C}}_{\mathcal{D}} \quad = \quad & R\{f_\alpha \mid \alpha \text{ has both target and source in } \mathcal{D}\} \\ & \oplus R\{f_\alpha \mid \alpha \text{ only has its target in } \mathcal{D}\} \\ & \oplus R\{f_\alpha \mid \text{other } \alpha\}.\end{aligned}$$

The first summand is exactly $(R\mathcal{D})^*$. The second summand is gone when $\mathcal{D}$ is an ideal in $\mathcal{C}$, and $R\mathcal{D}$ acts on the third summand as zero.

Similarly, $\downarrow^{\mathcal{C}}_{\mathcal{D}}$ preserves left injective $R\mathcal{C}$-modules if and only if $\mathcal{D}$ is R-co-taut.

4.3 Relative projectivity

Results in last section tell us when we can use the Eckmann-Shapiro Lemma. However, in order to obtain desired isomorphisms (i.e. reduction formulas) we need to know if and how we can express the involved modules as induced modules. In this section, we briefly go over the theory of relative projectivity, which will provide us a general framework to study the induced modules. Fixing a subcategory $\mathcal{D} \subset \mathcal{C}$, this theory relates certain $R\mathcal{C}$-modules with $R\mathcal{D}$-modules. When $\mathcal{D}$ is required to be full in the next section, we can get a very clear picture about this relationship. But in present section we just deal with arbitrary subcategories of $\mathcal{C}$ in order to give some results of the most generality. After establishing necessary results in the theory of relative projectivity, we will recall the Eckmann-Shapiro Lemma and give our first example of calculating higher limits, using the knowledge we obtain in and before this section.

Suppose A is an R-subalgebra of the R-algebra B. Let M be a B-module. Then there is a natural epimorphism $\epsilon : B \otimes_A M \to M$ given by the multiplication $\epsilon(f \otimes m) = fm$. We shall only consider the case of a category algebra $R\mathcal{C}$ with a subalgebra $R\mathcal{D}$, for some subcategory $\mathcal{D}$ of $\mathcal{C}$.

Definition 4.3.1. *Let M be an $R\mathcal{C}$-module. If the $R\mathcal{C}$-module epimorphism*

$$\epsilon = \epsilon_M : M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{C} \otimes_{R\mathcal{D}} M \to M$$

is split, then we say M is projective relative to $\mathcal{D}$, or simply $\mathcal{D}$-projective.

We have some equivalent descriptions of the relative projectivity of an $R\mathcal{C}$-module M.

Proposition 4.3.2. *Let $\mathcal{D} \subset \mathcal{C}$ be a subcategory and M an $R\mathcal{C}$-module. Then the following statements are equivalent:*

1. *The canonical surjective map $\epsilon : M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \to M$ splits;*
2. *M is a direct summand of $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$;*
3. *M is a direct summand of $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$, where N is an $R\mathcal{D}$-module.*
4. *if $0 \to M'' \to M' \to M \to 0$ is an exact sequence of $R\mathcal{C}$-modules which splits as an exact sequence of $R\mathcal{D}$-modules, then it splits as an exact sequence of $R\mathcal{C}$-modules.*

Proof. The statements 1, 2 and 4 are proved to be equivalent in a much more general context, see for instance [2], Chapter VI Proposition 3.6. When $\mathcal{C}$ is a finite group, statement 3 is well-known to be equivalent to the others, and the proof for category algebras is not seen in print even though the proof similar to that for group algebras.

We'll prove in this direction $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$. The proofs of $1 \Rightarrow 2 \Rightarrow 3$ are straightforward. Now we show $3 \Rightarrow 4$. Let M be a direct summand of $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ for some $R\mathcal{D}$-module N. Then we have two $R\mathcal{C}$-maps $\alpha : M \hookrightarrow N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ and $\beta : N \uparrow_{\mathcal{D}}^{\mathcal{C}} \to M$ satisfying the equality $\beta\alpha = 1_M$. Suppose $0 \to M'' \to M' \xrightarrow{\pi} M \to 0$ is an exact sequence of $R\mathcal{C}$-modules, which splits as an $R\mathcal{D}$-sequence. Then there is an $R\mathcal{D}$-map $\iota : M \downarrow_{\mathcal{D}}^{\mathcal{C}} \to M' \downarrow_{\mathcal{D}}^{\mathcal{C}}$ such that $\pi|_{R\mathcal{D}} \circ \iota = 1_{M \downarrow_{\mathcal{D}}^{\mathcal{C}}}$. We need to extend ι to an $R\mathcal{C}$-map ι', with the property that $\pi \circ \iota' = 1_M$. But first we want to construct an $R\mathcal{C}$-map from $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ to M'. In fact, since we have a split epimorphism $\beta|_{R\mathcal{D}} : N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} \to M \downarrow_{\mathcal{D}}^{\mathcal{C}}$ and a split monomorphism $\iota : M \downarrow_{\mathcal{D}}^{\mathcal{C}} \to M' \downarrow_{\mathcal{D}}^{\mathcal{C}}$, we can compose them to get an $R\mathcal{D}$-map $\iota \circ \beta|_{R\mathcal{D}} : N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} \to M' \downarrow_{\mathcal{D}}^{\mathcal{C}}$. By the canonical adjunction between $\uparrow_{\mathcal{D}}^{\mathcal{C}}$ and $\downarrow_{\mathcal{D}}^{\mathcal{C}}$, we obtain an $R\mathcal{C}$-map

$$\epsilon_{M'} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} (\iota \circ \beta_{R\mathcal{D}})) : (N \uparrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \to M',$$

corresponding to $\iota \circ \beta|_{R\mathcal{D}}$. Similarly let $\epsilon_{(N \uparrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} \gamma) : N \uparrow_{\mathcal{D}}^{\mathcal{C}} \to (N \uparrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ be the $R\mathcal{C}$-map corresponding to the natural $R\mathcal{D}$-map $\gamma : N \to (N \uparrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}}$, where $\gamma(g) = 1 \otimes 1 \otimes g$ for any $g \in N$. Then we get an $R\mathcal{C}$-map

$$\epsilon_{M'} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} (\iota \circ \beta|_{R\mathcal{D}})) \circ \epsilon_{(N\uparrow_{\mathcal{D}}^{\mathcal{C}})\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} \gamma) : N\uparrow_{\mathcal{D}}^{\mathcal{C}} \to M',$$

from which we construct the $R\mathcal{C}$-map $\iota' : M \hookrightarrow N\uparrow_{\mathcal{D}}^{\mathcal{C}} \to M'$ as

$$\epsilon_{M'} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} (\iota \circ \beta|_{R\mathcal{D}})) \circ \epsilon_{(N\uparrow_{\mathcal{D}}^{\mathcal{C}})\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}} \circ (R\mathcal{C} \otimes_{R\mathcal{D}} \gamma) \circ \alpha.$$

It's easy, though tedious, to check that $\pi \circ \iota' = 1_M$.

In the end we show $4 \Rightarrow 1$. Since ϵ_M is surjective, we can form an exact $R\mathcal{C}$-sequence $0 \to M'' \to M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}} \to M \to 0$, where M'' is the kernel of ϵ_M. This sequence splits after restriction to $R\mathcal{D}$, since there exists an $R\mathcal{D}$-map $\iota : M\downarrow_{\mathcal{D}}^{\mathcal{C}} \to (M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}})\downarrow_{\mathcal{D}}^{\mathcal{C}}$ sending $M\downarrow_{\mathcal{D}}^{\mathcal{C}}$ isomorphically onto $1 \otimes_{R\mathcal{D}} M\downarrow_{\mathcal{D}}^{\mathcal{C}}$, and $\epsilon_M|_{R\mathcal{D}} \circ \iota = 1_{M\downarrow_{\mathcal{D}}^{\mathcal{C}}}$ as $R\mathcal{D}$-maps. Thus the exact $R\mathcal{C}$-sequence splits by hypothesis, that is, $\epsilon_M : M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}} \to M$ splits. So we're done. □

Suppose M and N are two $R\mathcal{C}$-modules. Then we write $M|N$ if M is isomorphic to a direct summand of N. We collect some useful results below.

Proposition 4.3.3. *Let $\mathcal{C}$ be a finite EI-category. Then*

1. *if $\mathcal{E} \subset \mathcal{D}$ are subcategories of $\mathcal{C}$ and M is $\mathcal{E}$-projective then M is $\mathcal{D}$-projective;*
2. *if $\mathcal{E} \subset \mathcal{D}$ are subcategories of $\mathcal{C}$, N is an $R\mathcal{D}$-module which is $\mathcal{E}$-projective, and M is a direct summand of $N\uparrow_{\mathcal{D}}^{\mathcal{C}}$, then M is $\mathcal{E}$-projective;*
3. *if an $R\mathcal{C}$-module M is $\mathcal{D}$-projective, and it's projective as $R\mathcal{D}$-module, then M is a projective $R\mathcal{C}$-module.*

Proof. Since M is a direct summand of $M\downarrow_{\mathcal{E}}^{\mathcal{C}}\uparrow_{\mathcal{E}}^{\mathcal{C}}$ which can be written as $(M\downarrow_{\mathcal{E}}^{\mathcal{C}}\uparrow_{\mathcal{E}}^{\mathcal{D}})\uparrow_{\mathcal{D}}^{\mathcal{C}}$, we have $M|N\uparrow_{\mathcal{D}}^{\mathcal{C}}$ for an $R\mathcal{D}$-module. So M is $\mathcal{D}$-projective.

Next we prove 2. From $N|N\downarrow_{\mathcal{E}}^{\mathcal{D}}\uparrow_{\mathcal{E}}^{\mathcal{D}}$, we get

$$M \mid N\uparrow_{\mathcal{D}}^{\mathcal{C}} \mid (N\downarrow_{\mathcal{E}}^{\mathcal{D}}\uparrow_{\mathcal{E}}^{\mathcal{D}})\uparrow_{\mathcal{D}}^{\mathcal{C}} = (N\downarrow_{\mathcal{E}}^{\mathcal{D}})\uparrow_{\mathcal{E}}^{\mathcal{C}}.$$

It means M is $\mathcal{E}$-projective.

As for 3, the reason is that $M\downarrow_{\mathcal{D}}^{\mathcal{C}}$ must be a direct summand of $R\mathcal{D}^n$ for some positive cardinal n. Then $M\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}$ is a direct summand of $R\mathcal{D}^n\uparrow_{\mathcal{D}}^{\mathcal{C}} \cong R\mathcal{C}^n$. Thus M is a direct summand of a free $R\mathcal{C}$-module $R\mathcal{C}^n$. □

We need the following terminology before introducing our next two results.

Definition 4.3.4. *For each $R\mathcal{C}$-module M, we define the M-minimal objects to be those $x \in \operatorname{Ob}\mathcal{C}$ which satisfy the condition that $M(y) = 0$ if $y \not\cong x$ and* $\operatorname{Hom}(y, x)$ *is non-empty. Similarly we can define M-maximal objects.*

For example, the $\underline{R}$-minimal objects are the minimal objects of $\mathcal{C}$, and the $\underline{R}$-maximal objects are the maximal objects of $\mathcal{C}$. The $S_{x,V}$-minimal and $S_{x,V}$-maximal objects are the same: all $y \in [x]$.

Lemma 4.3.5. *If M is $\mathcal{D}$-projective: $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M \oplus M'$, then $\operatorname{Ob}\mathcal{D}$ contains all M-minimal objects, and if furthermore $\mathcal{D}$ is full then $M'(x) = 0$ for all $x \in \operatorname{Ob}\mathcal{D}$.*

Proof. If z is M-minimal and $z \notin \operatorname{Ob}\mathcal{D}$, then $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}(z) = \sum_{y \in \operatorname{Ob}\mathcal{D}} R\operatorname{Hom}(y, z) \otimes M(y) = 0$, since z is M-minimal. Therefore M cannot be a direct summand of $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}$ which is a contradiction. Hence $\operatorname{Ob}\mathcal{D}$ contains all M-minimal objects.

If $x \in \operatorname{Ob}\mathcal{D}$, then $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}(x) \cong M(x)$ when $\mathcal{D}$ is a full subcategory. It forces $M'(x) = 0$. □

We explain what's special about these M-minimal objects.

Lemma 4.3.6. *Suppose $\mathcal{D} \subset \mathcal{C}$ is a subcategory and M is an $R\mathcal{C}$-module which is $\mathcal{D}$-projective. Then $M(x)$ is $\operatorname{Aut}_{\mathcal{D}}(x)$-projective as an $R\operatorname{Aut}_{\mathcal{C}}(x)$-module, if x is M-minimal.*

Proof. We just evaluate the relation $M \bigm| M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}$ at x, and then the result follows. □

If $\underline{R}$ is $\mathcal{D}$-projective, then for any minimal object $x \in \operatorname{Ob}\mathcal{C}$, $\underline{R}(x) = R$ is $\operatorname{Aut}_{\mathcal{D}}(x)$-projective as an $R\operatorname{Aut}_{\mathcal{C}}(x)$-module. When $R = \mathbb{F}_p$ for some prime p dividing the order of $\operatorname{Aut}_{\mathcal{C}}(x)$, then $\operatorname{Aut}_{\mathcal{D}}(x)$ has to contain a Sylow p-subgroup of $\operatorname{Aut}_{\mathcal{C}}(x)$, by a standard result from the theory of vertices and sources for group algebras, see for instance Alperin [1] and Webb [41].

Note that $M(x)$ is not necessarily an indecomposable $R\operatorname{Aut}_{\mathcal{C}}(x)$-module even if M is indecomposable. The following is a simple example.

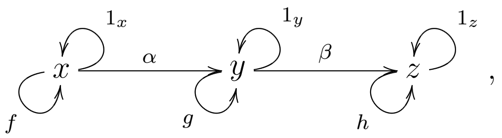
Example 4.3.7. *Let $R = \mathbb{F}_2$ and $\mathcal{C}$ the poset $x \to y$. Suppose M is the $R\mathcal{C}$-module given by $M(x) = R \oplus R$ and $M(y) = R$, where the unique non-isomorphism $M(x) \to M(y)$ is defined via $(a, b) \mapsto a+b$. Then it's easy to verify that M is an indecomposable module. Obviously $M(x) = R \oplus R$ is decomposable.*

It's time to give some examples for the theory of relative projectivity. The first example is taken from group representation theory.

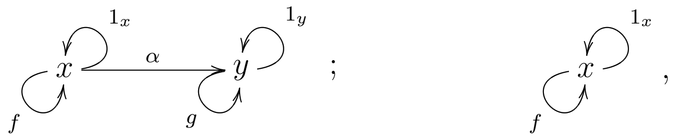
Example 4.3.8. *Let $\mathcal{C} = \hat{G}$ where G is a finite group. Suppose $\mathbb{F}_p$ is a field of characteristic p, a prime which divides the order of G. If M is an indecomposable $R\mathcal{C}$-module with vertex P, then M is projective relative to the subcategory $\mathcal{D} = \hat{P}$.*

The second example shows that an $R\mathcal{C}$-module can be projective relative to two distinct proper full subcategories of $\mathcal{C}$.

Example 4.3.9. *Let $\mathcal{C}$ be the following EI-category*



along with two full subcategories $\mathcal{D}$ and $\mathcal{E}$:



with $\alpha f = \alpha, g\alpha = \alpha, \beta g = \beta$ and $h\beta = \beta$. Let R be a commutative ring with an identity. We can diagrammatically describe the following three modules $\underline{R} \downarrow_{\mathcal{E}} \uparrow^{\mathcal{C}}$

$$R \xrightarrow{\alpha} (\alpha \otimes R) \xrightarrow{\beta} \beta\alpha \otimes R \, ,$$

$(\underline{R} \downarrow_{\mathcal{E}} \uparrow^{\mathcal{C}}) \downarrow_{\mathcal{D}}$

$$R \xrightarrow{\alpha} (\alpha \otimes R) \, ,$$

and $(\underline{R} \downarrow_{\mathcal{E}} \uparrow^{\mathcal{C}}) \downarrow_{\mathcal{D}} \uparrow^{\mathcal{C}}$

$$R \xrightarrow{\alpha} (\alpha \otimes R) \xrightarrow{\beta} (\beta\alpha \otimes R) \, .$$

So we have $\underline{R} \downarrow_{\mathcal{E}} \uparrow^{\mathcal{C}} = \underline{R}$ and $\underline{R} \downarrow_{\mathcal{D}} \uparrow^{\mathcal{C}} \cong \underline{R}$.

The main tool we will use for computing Ext groups $\mathrm{Ext}^*_{R\mathcal{C}}(M, N)$ is the Eckmann-Shapiro Lemma (Lemma 3.3.6). Here we content ourselves with the special form for a pair of algebras $R\mathcal{D} \subset R\mathcal{C}$. For convenience, we recall the lemma we stated in Section 3.3. Let M be $R\mathcal{C}$-module. If $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module then for any $R\mathcal{C}$-module N, we have

$$\mathrm{Ext}^*_{R\mathcal{C}}(M \uparrow_{\mathcal{D}}^{\mathcal{C}}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M, N \downarrow_{\mathcal{D}}^{\mathcal{C}}).$$

Here is a simple example showing us how the Eckmann-Shapiro Lemma works.

Example 4.3.10. *Suppose $\mathcal{C}$ is a finite poset with a unique minimal object m. We choose a subposet $\mathcal{D}$ containing only the minimal object m. Then it's easy to verify that $\mathcal{D}$ is R-co-taut since $R\mathcal{D} = R \cdot 1_m$, and furthermore $\underline{R}$ is $\mathcal{D}$-projective with $\underline{R} \uparrow_{\mathcal{D}}^{\mathcal{C}} \cong \underline{R}$. The Eckmann-Shapiro Lemma implies*

$$\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(R, N(m)).$$

Since $R\mathcal{D} \cong R$, we have $\mathrm{Ext}^0_{R\mathcal{D}}(R, N(m)) = N(m)$ and $\mathrm{Ext}^n_{R\mathcal{D}}(R, N(m)) = 0$ for all $n > 0$. It's not a surprise because $\underline{R} \cong P_{x,1}$ is projective.

In the next section, we will show every indecomposable $R\mathcal{C}$-module M is an induced module, i.e., of the form $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ for some subcategory $\mathcal{D}$ and an $R\mathcal{D}$-module N. Hence if $\mathcal{D}$ is R-co-taut, we can directly apply the Eckmann-Shapiro Lemma to get an isomorphism of Ext groups. In Section 4.6 we shall see that a certain class of EI-categories have some important homological properties determined by subcategories which are indeed posets. When such a category has a unique minimal object, the calculations of higher limits in the preceding example apply to it without any difficulties.

4.4 Vertices and sources

We continue to explore the ideas we presented in last section. From now on, we restrict our attention to the set of full subcategories of a fixed EI-category $\mathcal{C}$. We develop a theory of vertices and sources for category algebras, parallel to that for group algebras, which we describe now. For each indecomposable $R\mathcal{C}$-module M, we associate to it a full subcategory $\mathcal{V}_M$, which is unique and will be named the vertex of M. This subcategory $\mathcal{V}_M$ has the following important properties: $M \downarrow_{\mathcal{V}_M}$, which will be called the source for M, is an indecomposable $R\mathcal{V}_M$-module and satisfies $M \cong M \downarrow_{\mathcal{V}_M} \uparrow^{\mathcal{C}}$.

Since we can also prove $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ is indecomposable if N is an indecomposable $R\mathcal{D}$-module for a full subcategory $\mathcal{D}$. The properties of $\mathcal{V}_M$ allow us to parameterize all indecomposable $R\mathcal{C}$-module according to their vertices. Such a parametrization of indecomposable $R\mathcal{C}$-modules will be expressed in terms of a certain equivalence between two module categories. Since every indecomposable $R\mathcal{C}$-module is an induced module, we can get a variation of the Eckmann-Shapiro Lemma : $\mathrm{Ext}^*_{R\mathcal{C}}(M, N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M \downarrow_{\mathcal{D}}^{\mathcal{C}}, N \downarrow_{\mathcal{D}}^{\mathcal{C}})$, which in particular gives a formula for computing higher limits.

Let M be an indecomposable $R\mathcal{D}$-module that is $\mathcal{D}$-projective for some full subcategory $\mathcal{D}$. If $\mathcal{D}$ is a disjoint union of several connected components $\{\mathcal{D}_i\}_{i\in I}$, then from $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} = \oplus_{i\in I} M \downarrow_{\mathcal{D}_i}^{\mathcal{C}} \uparrow_{\mathcal{D}_i}^{\mathcal{C}}$ we know M is really projective relative to some $\mathcal{D}_i$. Indeed such a $\mathcal{D}_i$ has to be unique because it contains all the M-minimal objects. Thus for the rest of this chapter, when we talk about the $\mathcal{D}$-projectivity of some indecomposable $M \in R\mathcal{C}$-mod we usually mean that $\mathcal{D}$ is connected.

Proposition 4.4.1. *If M is $\mathcal{D}$-projective for a connected full subcategory $\mathcal{D} \subset \mathcal{C}$, then M is generated by its values on $\mathcal{D}$, that is, $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \cong M$.*

Proof. Suppose $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} = M' \oplus M''$ for some $R\mathcal{C}$-module M', M'' with $M' \cong M$ and $M''(x) = 0$ for all $x \in \mathrm{Ob}\,\mathcal{D}$. Let's take $y \notin \mathrm{Ob}\,\mathcal{D}$ and consider $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = M'(y) \oplus M''(y)$. We claim $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = \sum_{y > x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{R\mathcal{C}}(x, y) \otimes_{R\mathcal{D}} M(x)$ equals $M'(y)$. In fact $M'(x) = 1_{\mathcal{D}} \otimes M(x)$ for all $x \in \mathrm{Ob}\,\mathcal{D}$, and given any $\alpha \in \mathrm{Hom}(x, y)$, $\alpha \cdot M'(x) \subset M'(y)$, which means $\alpha \otimes M(x) \subset M'(y)$. When x and α run over all possible choices, we get exactly $\sum_{y > x \in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{R\mathcal{C}}(x, y) \otimes_{R\mathcal{D}} M(x) \subset M'(y)$ which is indeed an equality since the converse direction inclusion is certainly true. Thus the statement is correct. □

If $H \subset G$ are finite groups and M is an RG-module which is H-projective, then we usually have $M \downarrow_H \uparrow^G \cong M \oplus M'$, with $M' \neq 0$ in general. We can explain the difference between this and our result by the fact that, regarded as categories, $\hat{H}$ isn't a full subcategories of $\hat{G}$ (i.e. $\hat{H} \neq \hat{G}$) in general.

If M is $\mathcal{D}$-projective ($\mathcal{D}$ full), then the natural surjection $\epsilon : M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \to M$ is an isomorphism. Let $y \in \operatorname{Ob}\mathcal{C}$. From $\epsilon_y : M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) \xrightarrow{\cong} M(y)$ we get

$$\begin{aligned}\epsilon_y(M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}(y)) &= \epsilon_y(\textstyle\sum_{x \in \operatorname{Ob}\mathcal{D}_{\leq y}} R\operatorname{Hom}_{\mathcal{C}}(x,y) \otimes_{R\mathcal{D}} M(x)) \\ &= \textstyle\sum_{x \in \operatorname{Ob}\mathcal{D}_{\leq y}} R\operatorname{Hom}_{\mathcal{C}}(x,y) \cdot M(x) \\ &= M(y).\end{aligned}$$

This implies the $R\mathcal{C}$-module M is generated by its values on objects in $\mathcal{D}$. However, $\sum_{x \in \operatorname{Ob}\mathcal{D}_{\leq y}} R\operatorname{Hom}_{\mathcal{C}}(x,y) \cdot M(x) = M(y)$ for any $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$ doesn't guarantee M is $\mathcal{D}$-projective. We can consider the category $x \rightrightarrows y$ with two non-isomorphisms and two trivial isomorphisms. Then the trivial module $\underline{R}$ is projective relative to the whole category, not $\{x\}$ although both non-isomorphisms send $\underline{R}(x) = R$ isomorphically to $\underline{R}(y) = R$.

Proposition 4.4.1 gives us some hints about the existence of the next two important results.

Theorem 4.4.2. *Let $\mathcal{D} \subset \mathcal{C}$ be a (connected) full subcategory and N an indecomposable $R\mathcal{D}$-module. Then the $R\mathcal{C}$-module $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ is indecomposable and moreover is $\mathcal{D}$-projective.*

Proof. Suppose $N \uparrow_{\mathcal{D}}^{\mathcal{C}} = N_1 \oplus N_2$, where N_1, N_2 are both non-zero. Then $N = N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} = N_1 \downarrow_{\mathcal{D}}^{\mathcal{C}} \oplus N_2 \downarrow_{\mathcal{D}}^{\mathcal{C}}$, and since N is indecomposable we must have $N_1 \downarrow_{\mathcal{D}}^{\mathcal{C}} = N$ and $N_2 \downarrow_{\mathcal{D}}^{\mathcal{C}} = 0$ (or the other way around). Now that $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ is generated by its values on $\mathcal{D}$ implies $N_2 = 0$. Hence inductively we can show $N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ is indecomposable, and its $\mathcal{D}$-projectivity follows from Definition 4.3.1. □

One can compare the above theorem with the Green's indecomposability theorem in group representation theory (see for instance Alperin [1] or Benson [4]).

Definition 4.4.3. *Let x be an object of an EI-category $\mathcal{C}$ and $H \subset \operatorname{Aut}_{\mathcal{C}}(x)$. Then we use $\{x\}_H$ to denote the subcategory of $\mathcal{C}$ whose object is x and morphisms are elements of H. If $H = \operatorname{Aut}_{\mathcal{C}}(x)$ then we will use $\{x\}$, instead of $\{x\}_{\operatorname{Aut}_{\mathcal{C}}(x)}$, to denote the full subcategory of $\mathcal{C}$ with a single object x. We use $\{[x]\}$ to denote the full subcategory of $\mathcal{C}$ consisting of all objects which are isomorphic to x.*

Given an x, one can choose the full subcategory $\{[x]\}$ and use an indecomposable $R\{[x]\}$-module N to generate an $R\mathcal{C}$-module $N \uparrow_{\{[x]\}}^{\mathcal{C}}$. Then Theorem 4.4.2 asserts that such an induced module is indecomposable. This implies that $R\mathcal{C}$ is not of finite representation type if, for some $x \in \mathrm{Ob}\,\mathcal{C}$, $R\,\mathrm{Aut}_{\mathcal{C}}(x)$ is not. There is a theorem of Higman says that if $R = \mathbb{F}_p$ and G is a finite group then $\mathbb{F}_p G$ has finite representation type if and only if the Sylow p-subgroups of G are cyclic, see Webb [41].

Proposition 4.4.4. *Let M be an indecomposable $R\mathcal{C}$-module which is $\mathcal{D}$-projective for a connected full subcategory $\mathcal{D} \subset \mathcal{C}$. Then $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ is indecomposable.*

Proof. Suppose $M \downarrow^{\mathcal{C}}_{\mathcal{D}} = M_1 \oplus \cdots \oplus M_n$ is a direct sum of indecomposable $R\mathcal{D}$-modules. Then $M \cong M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} = M_1 \uparrow^{\mathcal{C}}_{\mathcal{D}} \oplus \cdots \oplus M_n \uparrow^{\mathcal{C}}_{\mathcal{D}}$. Since M is indecomposable, we have $M | M_i \uparrow^{\mathcal{C}}_{\mathcal{D}}$ for some index i. This implies $M(x) | M_i \uparrow^{\mathcal{C}}_{\mathcal{D}} (x) = M_i(x)$ for all $x \in \mathrm{Ob}\,\mathcal{D}$, hence $M \downarrow^{\mathcal{C}}_{\mathcal{D}} = M_i$ is indecomposable. □

The preceding two results give us an equivalence of two module categories.

Definition 4.4.5. *Let $\mathcal{D} \subset \mathcal{C}$ be a connected full subcategory. We define $R\mathcal{C}_{\mathcal{D}}$-mod to be the full subcategory of $R\mathcal{C}$-mod consisting of all $\mathcal{D}$-projective $R\mathcal{C}$-modules.*

For the sake of simplicity, we write $\mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M, N)$ for the set of morphisms between two modules $M, N \in R\mathcal{C}_{\mathcal{D}}$-mod.

Proposition 4.4.6. *The functor $\downarrow^{\mathcal{C}}_{\mathcal{D}} : R\mathcal{C}_{\mathcal{D}}\text{-mod} \to R\mathcal{D}\text{-mod}$ is an equivalence with $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ as its inverse.*

Proof. From the previous results we see the two functors are well-defined on objects, while $\downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Id}_{\mathcal{C}_{\mathcal{D}}}$ and $\uparrow^{\mathcal{C}}_{\mathcal{D}} \downarrow^{\mathcal{C}}_{\mathcal{D}} \cong \mathrm{Id}_{\mathcal{D}}$. The actions of induction and restriction on the morphisms are very clear. Furthermore on morphisms we have the following isomorphisms

$$\mathrm{Hom}_{R\mathcal{D}}(M \downarrow^{\mathcal{C}}_{\mathcal{D}}, N \downarrow^{\mathcal{C}}_{\mathcal{D}}) \cong \mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}, N) \cong \mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M, N),$$

and

$$\mathrm{Hom}_{R\mathcal{C}_{\mathcal{D}}}(M \uparrow^{\mathcal{C}}_{\mathcal{D}}, N \uparrow^{\mathcal{C}}_{\mathcal{D}}) \cong \mathrm{Hom}_{R\mathcal{D}}(M, N \uparrow^{\mathcal{C}}_{\mathcal{D}} \downarrow^{\mathcal{C}}_{\mathcal{D}}) \cong \mathrm{Hom}_{R\mathcal{D}}(M, N).$$

So $\downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}}$ and $\uparrow^{\mathcal{C}}_{\mathcal{D}} \downarrow^{\mathcal{C}}_{\mathcal{D}}$ are also identities on morphisms, because both M and N are generated by their values on $\mathcal{D}$. □

Note that the second isomorphism doesn't depend on the relative projectivity of the modules, since we always have $\mathrm{Hom}_{R\mathcal{C}}(M\uparrow_{\mathcal{D}}^{\mathcal{C}}, N\uparrow_{\mathcal{D}}^{\mathcal{C}}) \cong \mathrm{Hom}_{R\mathcal{D}}(M, N)$, as long as $\mathcal{D} \subset \mathcal{C}$ is a full subcategory.

Now we're ready to develop a theory of vertices and sources for category algebras. The next proposition and its corollary suggest that *we should restrict our attention to the subcategories which are ideals in* $\mathcal{C}$, in order to give a proper definition of the vertex of an indecomposable module.

Proposition 4.4.7. *Suppose $\mathcal{D}$ and $\mathcal{E}$ are full subcategories of $\mathcal{C}$ whose objects form ideals in $\mathrm{Ob}\,\mathcal{C}$. Suppose M is an indecomposable $R\mathcal{C}$-module. Then $M\downarrow_{\mathcal{D}}\uparrow^{\mathcal{C}}\downarrow_{\mathcal{E}}\uparrow^{\mathcal{C}} \cong M\downarrow_{\mathcal{D}\cap\mathcal{E}}\uparrow^{\mathcal{C}}$.*

Proof. We need to consider the structure of ${}_{R\mathcal{E}}R\mathcal{C}\bigotimes_{R\mathcal{D}}M$. Since $\mathrm{Ob}\,\mathcal{E}$ forms an ideal in $\mathrm{Ob}\,\mathcal{C}$, we get $R\mathcal{C} \cong R\mathcal{E}$ as an $R\mathcal{E}$-module. The only terms in this direct sum on which $\mathcal{D}$ is non-zero in the action from the right are the ones where x is in $\mathrm{Ob}\,\mathcal{D}$. Regarded as a right $R\mathcal{D}$-module, $R\mathcal{C}$ can be identified with $R\mathcal{D} = \bigoplus_{x\in\mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}_{\mathcal{C}}(x,-)$. So as an $R\mathcal{E}$-$R\mathcal{D}$-bimodule, $R\mathcal{C} \cong \bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})} R\,\mathrm{Hom}_{\mathcal{C}}(x,-)$. Thus

$$\begin{aligned} R\mathcal{C}\bigotimes_{R\mathcal{E}}R\mathcal{C}\bigotimes_{R\mathcal{D}}M &\cong R\mathcal{C}\bigotimes_{R\mathcal{E}}\{\bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})}R\mathrm{Hom}_{\mathcal{C}}(x,-)\}\bigotimes_{R\mathcal{D}}M \\ &= R\mathcal{C}\bigotimes_{R\mathcal{E}}\{\bigoplus_{x\in\mathrm{Ob}(\mathcal{D}\cap\mathcal{E})}R\mathrm{Hom}_{\mathcal{C}}(x,-)\}\bigotimes_{R(\mathcal{D}\cap\mathcal{E})}M \\ &\cong R\mathcal{C}\bigotimes_{R\mathcal{E}}R\mathcal{E}\bigotimes_{R(\mathcal{D}\cap\mathcal{E})}M \\ &\cong R\mathcal{C}\bigotimes_{R(\mathcal{D}\cap\mathcal{E})}M. \end{aligned}$$

□

We note that this argument does not work for an arbitrary pair of full (R-taut, R-co-taut etc) subcategories relative to which M is projective.

Corollary 4.4.8. *Suppose $\mathcal{D}$ and $\mathcal{E}$ are full subcategories of $\mathcal{C}$ which are ideals. Let M be an indecomposable $R\mathcal{C}$-module, which is both $\mathcal{D}$-projective and $\mathcal{E}$-projective. Then M is also $\mathcal{D}\cap\mathcal{E}$-projective. Thus for any indecomposable $R\mathcal{C}$-module M, there exists a unique smallest full subcategory $\widetilde{\mathcal{V}_M}$, among all ideals in $\mathcal{C}$, relative to which M is projective.*

Proof. We just need to check that $\mathcal{D}\cap\mathcal{E}$ forms an ideal in $\mathcal{C}$, and then the results follow from the above lemma. □

Obviously, $\widetilde{\mathcal{V}_M}$ has to be connected, because if $\widetilde{\mathcal{V}_M} = \mathcal{D}_1 \cup \mathcal{D}_2$, then $M \downarrow^{\mathcal{C}}_{\widetilde{\mathcal{V}_M}} = M \downarrow^{\mathcal{C}}_{\mathcal{D}_1} \oplus M \downarrow^{\mathcal{C}}_{\mathcal{D}_2}$, and M must be projective relative to one of its connected components, which contradicts with the minimality of $\widetilde{\mathcal{V}_M}$.

At this point, one who is familiar with the definition of the vertex for an indecomposable group module might want to define the category $\widetilde{\mathcal{V}_M}$ to be the "vertex" of M. However, there is a possibility to throw away some objects $x \in \widetilde{\mathcal{V}_M}$ for which $M(x) = 0$, and subsequently obtain a category smaller than $\widetilde{\mathcal{V}_M}$ for M, relative to which M is still projective.

Definition 4.4.9. *For any $R\mathcal{C}$-module M we define the full subcategory of $\mathcal{C}$, $\mathcal{C}_M$ to be a category consisting of all $y \in \operatorname{Ob}\mathcal{C}$ with $\operatorname{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$ for some M-minimal object x. Similarly we define $\mathcal{C}^M$ to be the full subcategory containing all $z \in \operatorname{Ob}\mathcal{C}$ with $\operatorname{Hom}(z, x) \neq \emptyset$ for some M-maximal object x. In particular, we have $\mathcal{C}_{S_{x,V}} = \mathcal{C}_{\geq x}$ and $\mathcal{C}^{S_{x,V}} = \mathcal{C}_{\leq x}$.*

In fact, $\mathcal{C}_M$ is a co-ideal in $\mathcal{C}$ generated by the M-minimal objects, and $\mathcal{C}^M$ is an ideal in $\mathcal{C}$ generated by the M-maximal objects. In particular, we have $\mathcal{C}^{\underline{R}} = \mathcal{C}_{\underline{R}} = \mathcal{C}$.

Definition 4.4.10. *The full subcategory $\mathcal{V}_M = \widetilde{\mathcal{V}_M} \cap \mathcal{C}_M$ is called the vertex of M.*

Note that $\mathcal{V}_M$ is connected but is not necessarily an ideal in $\mathcal{C}$, though it's an ideal in $\mathcal{C}_M$. Also note that by Proposition 4.4.1, every indecomposable $R\mathcal{C}$-module is an induced module. Next we want to get an alternative description of the vertex of M, free of the use of $\mathcal{C}_M$.

Definition 4.4.11. *Let $\mathcal{D} \subset \mathcal{C}$ be a subcategory. Then $\mathcal{D}$ is said to be convex if whenever there is a sequence of morphisms $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathcal{C}$ with $x, z \in \operatorname{Ob}\mathcal{D}$, then both α and β are in $\operatorname{Mor}(\mathcal{D})$.*

Ideals and co-ideals in $\mathcal{C}$ are full convex subcategories. Let M be an indecomposable $R\mathcal{C}$-module. Then its vertex $\mathcal{V}_M$ is convex. Note that in general a convex subcategory $\mathcal{D}$ doesn't have to be full. Since the intersection of two convex subcategories is still convex, if not empty, it's natural to define the *convex hull* of a subcategory $\mathcal{D}$ of $\mathcal{C}$ as the smallest convex subcategory containing $\mathcal{D}$.

Let's make a detour to get an interesting property of convex subcategories before we return to the developing of our own theory. When we have a short exact sequence of groups $1 \to N \to G \to H \to 1$. Then any RH-module can be lifted to an RG-module via the projection. But in general an RN-module cannot be regarded as an RG-module. From next result, it seems the reason is that $\hat{N}$ is not convex in $\hat{G}$ (actually the only convex subcategory of $\hat{G}$ is itself).

Proposition 4.4.12. *Let $\mathcal{D} \subset \mathcal{C}$ be a convex subcategory. Then there exists a functor $\tau_{\mathcal{D}}$ from $R\mathcal{D}$-mod to $R\mathcal{C}$-mod.*

Proof. Suppose $M \in R\mathcal{D}$-mod. Let $\tau_{\mathcal{D}}(M) = M$ as R-modules. We shall define an $R\mathcal{C}$-module structure on $\tau_{\mathcal{D}}(M)$. For any $\alpha \in \mathrm{Mor}(\mathcal{C})$, we require $\alpha \cdot \tau_{\mathcal{D}}(M) = 0$ unless $\alpha \in \mathrm{Mor}(\mathcal{D})$. This gives a well-defined $R\mathcal{C}$-module structure on $\tau_{\mathcal{D}}(M)$ because for any $\alpha, \beta \in \mathrm{Mor}(\mathcal{C})$ we must have $(\beta\alpha) \cdot \tau_{\mathcal{D}}(M) = \beta \cdot (\alpha \cdot \tau_{\mathcal{D}}(M))$ since $\mathcal{D}$ is convex. The action of $\tau_{\mathcal{D}}$ on homomorphisms is the same as the identity. □

The above result give us a different parametrization of $R\mathcal{C}$-modules through the convex hull of the support of each $R\mathcal{C}$-module. Note that the condition "convex" cannot be dropped. Let's take $\mathcal{C}$ to be a poset of three objects $x \longrightarrow y \longrightarrow z$ and choose the subposet $\mathcal{D} : x \longrightarrow z$. Then the trivial $R\mathcal{D}$-module $\underline{R}$ is not an $R\mathcal{C}$-module. We should warn readers that the vertex of $\tau_{\mathcal{V}_M}(M \downarrow^{\mathcal{C}}_{\mathcal{V}_M})$ is usually larger than $\mathcal{V}_M$, which means $\tau_{\mathcal{V}_M}(M \downarrow^{\mathcal{C}}_{\mathcal{V}_M})$ is not $\mathcal{V}_M$-projective in general. One can jump to Propositions 4.4.18 and 4.4.19 for some examples (where we give the vertices of indecomposable projective and atomic modules).

Proposition 4.4.13. *Let M be an indecomposable $R\mathcal{C}$-module and $\mathcal{D}$ a full subcategory of $\mathcal{C}$. Then the following statements are equivalent:*

1. *$\mathcal{D}$ is the vertex of M;*
2. *$\mathcal{D}$ is the smallest ideal in $\mathcal{C}_M$, relative to which M is projective;*
3. *$\mathcal{D}$ is the smallest full convex subcategory of $\mathcal{C}$, relative to which M is projective.*

Proof. $1 \Rightarrow 2$: If $\mathcal{D} = \mathcal{V}_M$, then by definition $\mathcal{D}$ is full convex and M is $\mathcal{D}$-projective. Suppose $\mathcal{E}$ is an ideal in $\mathcal{C}_M$, relative to which M is projective. We claim $\mathcal{D} \subset \mathcal{E}$. In fact, $\mathcal{D}$ is an ideal in $\mathcal{C}_M$, and so is $\mathcal{D} \cap \mathcal{E}$. We can naturally extend $\mathcal{D} \cap \mathcal{E}$ to an ideal $\widetilde{\mathcal{D} \cap \mathcal{E}} \subset \widetilde{\mathcal{V}_M}$ in $\mathcal{C}$, relative to which M is projective. But then by definition we have $\widetilde{\mathcal{V}_M} = \widetilde{\mathcal{D} \cap \mathcal{E}}$, which implies $\mathcal{D} = \widetilde{\mathcal{V}_M} \cap \mathcal{C}_M = \widetilde{\mathcal{D} \cap \mathcal{E}} \cap \mathcal{C}_M = \mathcal{D} \cap \mathcal{E}$.

$2 \Rightarrow 3$: Let $\mathcal{E}$ be a full convex subcategory for which M is $\mathcal{E}$-projective. Then $\mathcal{E}$ contains all M-minimal objects, and thus $\mathcal{E} \cap \mathcal{C}_M$ must be an ideal in $\mathcal{C}_M$. Since as an $R\mathcal{C}_M$-module M is projective relative to $\mathcal{E} \cap \mathcal{C}_M$, we have $\mathcal{D} \subset \mathcal{E}$.

$3 \Rightarrow 1$: Let $\mathcal{E}$ be an ideal in $\mathcal{C}$, relative to which M is projective. Then $\mathcal{E} \cap \mathcal{C}_M$ is a full convex subcategory in $\mathcal{C}$, which means $\mathcal{D} \subset \mathcal{E} \cap \mathcal{C}_M$. We can take $\mathcal{E}$ to be $\widetilde{\mathcal{V}_M}$, and this results in an inclusion $\mathcal{D} \subset \mathcal{V}_M$, which can be shown to be an equality by extend $\mathcal{D}$ to an ideal in $\widetilde{\mathcal{V}_M}$. □

The term "convex" in part 3 of above proposition can not be removed, see Remark 4.4.15 below.

Proposition 4.4.14. *Let $\mathcal{D}$ be a connected full subcategory of $\mathcal{C}$ and N an indecomposable $R\mathcal{D}$-module with the vertex $\mathcal{V}_N \subset \mathcal{D}$. Then the indecomposable $R\mathcal{C}$-module $M = N \uparrow_{\mathcal{D}}^{\mathcal{C}}$ is $\mathcal{V}_N$-projective. If $\mathcal{V}_N$ is a (connected and full) convex subcategory of $\mathcal{C}$, then $\mathcal{V}_M = \mathcal{V}_N$.*

If M is an indecomposable $R\mathcal{C}$-module whose vertex is $\mathcal{V}_M$, and $\mathcal{D}$ is a connected full subcategory containing $\mathcal{V}_M$, then $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$ is an indecomposable $R\mathcal{D}$-module whose vertex is $\mathcal{V}_M$.

Proof. The first statement makes sense because of Theorem 4.4.2, and it is by Proposition 4.3.3 (2) we know $\mathcal{V}_M \subset \mathcal{V}_N$. After we prove the second part, we can show $\mathcal{V}_M$ is exactly $\mathcal{V}_N$, if $\mathcal{V}_N$ is a convex subcategory of $\mathcal{C}$.

The second statement makes sense because of Proposition 4.4.4. Suppose $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$ has vertex $\mathcal{E}' \subset \mathcal{D}$. Since $M \cong (M \downarrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{V}_M}^{\mathcal{D}} \uparrow_{\mathcal{V}_M}^{\mathcal{C}} \cong (M \downarrow_{\mathcal{V}_M}^{\mathcal{C}} \uparrow_{\mathcal{V}_M}^{\mathcal{D}}) \uparrow_{\mathcal{D}}^{\mathcal{C}}$, we obtain $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \cong M \downarrow_{\mathcal{V}_M}^{\mathcal{C}} \uparrow_{\mathcal{V}_M}^{\mathcal{D}} = (M \downarrow_{\mathcal{D}}^{\mathcal{C}}) \downarrow_{\mathcal{V}_M}^{\mathcal{D}} \uparrow_{\mathcal{V}_M}^{\mathcal{D}}$. Hence $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$ is $\mathcal{V}_M$-projective and $\mathcal{E}' \subset \mathcal{V}_M$. But by the first part, $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ has a vertex that is contained in $\mathcal{E}'$. Since $M \downarrow_{\mathcal{D}}^{\mathcal{C}} \uparrow_{\mathcal{D}}^{\mathcal{C}} \cong M$, we must have $\mathcal{V}_M \subset \mathcal{E}'$.

Now we go back to finish proving part 1. Let $\mathcal{V}_M$ be the vertex of $M = N \uparrow_{\mathcal{D}}^{\mathcal{C}}$, from part 2 and $M \downarrow_{\mathcal{D}}^{\mathcal{C}} = N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}}$ we get desired equality $\mathcal{V}_M = \mathcal{V}_N$ because $M \downarrow_{\mathcal{D}}^{\mathcal{C}}$ still has vertex $\mathcal{V}_M$ and $N \uparrow_{\mathcal{D}}^{\mathcal{C}} \downarrow_{\mathcal{D}}^{\mathcal{C}} = N$ has vertex $\mathcal{V}_N$. □

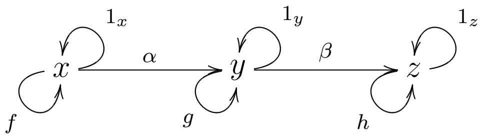
Remark 4.4.15. *In the first part of Proposition 4.4.14, if $\mathcal{V}_N$ is not a convex subcategory in $\mathcal{C}$, then it's not necessarily true that $\mathcal{V}_M = \mathcal{V}_N$, where $M = N \uparrow_{\mathcal{D}}^{\mathcal{C}}$. One can check Example 4.5.8 in the next section, where we have a pair of categories $\mathcal{D} \subset \mathcal{C}$. If we choose the $R\mathcal{D}$-module $N = \underline{R}$, then $\mathcal{V}_N = \mathcal{D}$, which is not convex in $\mathcal{C}$. The* *induced module* $M = N\uparrow_{\mathcal{D}}^{\mathcal{C}} = \underline{R}\uparrow_{\mathcal{D}}^{\mathcal{C}}$ *is isomorphic to the trivial* $R\mathcal{C}$*-module* $\underline{R}$*, which is* $\mathcal{V}_N$*-projective and whose vertex is shown to be* $\mathcal{V}_M = \mathcal{C}$*.*

Now we define the source. Since $M\downarrow_{\mathcal{V}_M}^{\mathcal{C}}\uparrow_{\mathcal{V}_M}^{\mathcal{C}}\cong M$ and $M\downarrow_{\mathcal{V}_M}^{\mathcal{C}}$ is indecomposable, M is determined up to isomorphism by the indecomposable $R\mathcal{V}_M$-module $M\downarrow_{\mathcal{V}_M}^{\mathcal{C}}$.

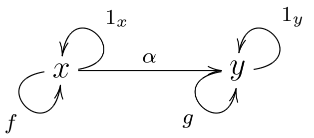
Definition 4.4.16. *Suppose* M *is an indecomposable* $R\mathcal{C}$*-module with the vertex* $\mathcal{V}_M$*. Then* $M\downarrow_{\mathcal{V}_M}$ *is called the source for* M*.*

It's time to give some examples, which can be helpful for understanding the established results.

Example 4.4.17. *Recall in Example 4.3.9 we had the category* $\mathcal{C}$



and two full subcategories $\mathcal{D}$ *and* $\mathcal{E}$*:*



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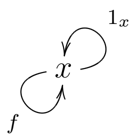

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with $\alpha f = \alpha, g\alpha = \alpha, \beta g = \beta$ *and* $h\beta = \beta$*. Both* $\mathcal{D}$ *and* $\mathcal{E}$ *are ideals in* $\mathcal{C}$ *and we showed* $\underline{R}\downarrow_{\mathcal{E}}\uparrow^{\mathcal{C}} = (\underline{R}\downarrow_{\mathcal{E}}\uparrow^{\mathcal{C}})\downarrow_{\mathcal{D}}\uparrow^{\mathcal{C}}\cong \underline{R}$*. Obviously the vertex of* $\underline{R}$ *is* $\mathcal{V}_{\underline{R}} = \mathcal{E}$*, and the source for it is the* $R\mathcal{E}$*-module* $\underline{R}$*.*

Next we provide two more examples, which give us the vertices and sources for two kinds of important $R\mathcal{C}$-modules: indecomposable projectives and indecomposable atomic modules (e.g. simple modules). Recall that we denote by $\{[x]\}$ the full subcategory of $\mathcal{C}$ (equivalent to $\widehat{\mathrm{Aut}_{\mathcal{C}}(x)}$) consisting of objects in the isomorphism class $[x]$.

Proposition 4.4.18. *The vertex of the indecomposable projective module* $P_{x,V}$ *is* $\{[x]\}$*. The source for* $P_{x,V}$ *is* $P_V = P_{x,V}\downarrow_{[x]}^{\mathcal{C}}$*, the projective cover of* V *as an* $R\{[x]\}$*-module.*

Proof. For simplicity, we can assume $\mathcal{C}$ is skeletal. Then we can easily check that $P_{x,V} \downarrow^{\mathcal{C}}_{\mathcal{C}_{\le x}} \uparrow^{\mathcal{C}}_{\mathcal{C}_{\le x}} = P_V \uparrow^{\mathcal{C}}_{\mathcal{C}_{\le x}} = R\mathcal{C} \otimes_{R\mathcal{C}_{\le x}} R\,\mathrm{Aut}(x) \cdot e_{x,V} = R\mathcal{C} \otimes_{R\mathcal{C}_{\le x}} e_{x,V} \cong P_{x,V}$. The isomorphism is defined via multiplication $\alpha \otimes e_{x,V} \mapsto \alpha \cdot e_{x,V}$, which is obviously surjective and is indeed injective since $e_{x,V}$ is an idempotent. Now $\mathcal{C}_{P_{x,V}} = \mathcal{C}_{\ge x}$. So by definition the vertex of $P_{x,V}$ is $\mathcal{C}_{\le x} \cap \mathcal{C}_{\ge x} = \{x\}$. $\square$

Recall that a module M is called atomic if $M(y)(= 1_y \cdot M) \neq 0$ if and only if $y \cong x$ for an object $x \in \mathrm{Ob}\,\mathcal{C}$. For such a module M, it is indecomposable if and only if $M(x)$ is indecomposable as an $R\,\mathrm{Aut}(x)$-module.

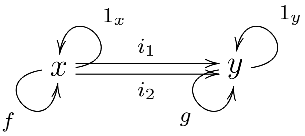
Proposition 4.4.19. *Let M be an indecomposable atomic module concentrated on $[x] \subset \mathrm{Ob}\,\mathcal{C}$. Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}_{\ge x}$ whose object set consists of $[x]$ and those $y \ncong x$ which satisfy the condition that $\mathrm{Irr}_{\mathcal{C}}(x,y) \neq \emptyset$. Then M is $\mathcal{D}$-projective, and $\mathcal{V}_M$ is the smallest convex full subcategory that contains $\mathcal{D}$. The source for M is itself (but regarded as an $R\mathcal{V}_M$-module).*

Proof. It's easy to verify that $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M$ and there is no proper full subcategory of $\mathcal{D}$ having the same property, because if $y \in \mathrm{Ob}\,\mathcal{C}_{\ge x}$, $\mathrm{Irr}_{\mathcal{C}}(x,y) \neq \emptyset$ and $y \notin \mathrm{Ob}\,\mathcal{D}$ then $0 \neq R\,\mathrm{Irr}_{\mathcal{C}}(x,y) \otimes_{R\mathcal{D}} M(x) \subset R\,\mathrm{Hom}(x,y) \otimes_{R\mathcal{D}} M(x) \subset M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} (y)$, which contradicts with the fact $M \downarrow^{\mathcal{C}}_{\mathcal{D}} \uparrow^{\mathcal{C}}_{\mathcal{D}} (y) \cong M(y) = 0$.

From above, we see $\mathcal{D} \subset \mathcal{V}_M$. By 4.4.13 (3), $\mathcal{V}_M$ is exactly the smallest full convex subcategory containing $\mathcal{D}$. $\square$

The last example first discusses the vertex and source for an indecomposable module other than indecomposable projective or atomic modules, and then describes the representation type of $R\mathcal{C}$.

Example 4.4.20. *Given a category $\mathcal{C}$*



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with $i_1 f = i_1, i_2 f = i_2, g i_1 = i_2$ and $g i_2 = i_1$. Let $R = \mathbb{F}_2$ be a field of characteristic 2. We consider the indecomposable module M such that $M(x) = \mathbb{F}_2$ and $M(y) = \mathbb{F}_2 \oplus \mathbb{F}_2$. The maps i_1, i_2 send $M(x) = \mathbb{F}_2$ to the first and the second component, respectively, of $M(y) = \mathbb{F}_2 \oplus \mathbb{F}_2$, and g interchanges the two entries of $\mathbb{F}_2 \oplus \mathbb{F}_2$ (it's easy to verify *these define a functor $M : \mathcal{C} \to R$-mod, and it gives rise to a $\mathbb{F}_2\mathcal{C}$-module which isn't projective or simple).*

We can see M is an induced module, which is not projective or simple. It has $\mathcal{D} = \{x\} \cong \widehat{\mathrm{Aut}(x)}$ as its vertex with source $M(x) = \mathbb{F}_2$, since $\mathbb{F}_2 \uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = (i_1 \otimes \mathbb{F}_2) \oplus (i_2 \otimes \mathbb{F}_2)$ as $\mathbb{F}_2 \mathrm{Aut}(y)$-module.

This category $\mathcal{C}$ has exactly three full subcategories: $\{x\}, \{y\}$ and $\mathcal{C}$ itself, all of which are connected and convex. By Proposition 4.4.6, the indecomposable $R\mathcal{C}$-modules can be put into three distinct classes, each of them is determined by one of the three subcategories. If we consider $\{x\}$, then any indecomposable $R\mathcal{C}$-module, which is $\{x\}$-projective, is generated by its value on x and corresponds uniquely to an indecomposable $\mathbb{F}_2 \mathrm{Aut}_{\mathcal{C}}(x) (\cong \mathbb{F}_2 C_2)$-module. There are finitely many of them, up to isomorphism, because any indecomposable $\mathbb{F}_2 C_2$-modules is isomorphic to either $\mathbb{F}_2 C_2$ or $\mathbb{F}_2$, and $R\mathcal{C} \otimes_{R \mathrm{Aut}_{\mathcal{C}}(x)} -$ is exact. Similar statement can be posed for indecomposable $R\mathcal{C}$-modules which are $\{y\}$-projective. There exist modules which are not projective relative to either of the two proper subcategories of $\mathcal{C}$. The typical ones are the atomic modules concentrated on x, e.g., $S_{x,1}$. However, there are modules other than the atomic ones. For instance, $\underline{\mathbb{F}_2}$ has vertex $\mathcal{V}_{\underline{\mathbb{F}_2}} = \mathcal{C}$. In general if M has vertex $\mathcal{C}$ then $M(x) \neq \{0\}$ because otherwise it'll be atomic, concentrated on y with vertex $\{y\}$. We show $\mathbb{F}_2\mathcal{C}$ has infinite representation type by constructing infinitely many non-isomorphic modules whose vertices are $\mathcal{C}$. Define for any $n \in N$ an (indecomposable) $\mathbb{F}_2\mathcal{C}$-module M_n such that $M_n(x) = [\mathbb{F}_2 \mathrm{Aut}_{\mathcal{C}}(x)]^n$ and $M(y) = \mathbb{F}_2 i_1 + \mathbb{F}_2 i_2$. It's certainly not $\{y\}$-projective, and is not $\{x\}$-projective when $n > 1$ because $[R\mathcal{C} \otimes_{R \mathrm{Aut}_{\mathcal{C}}(x)} M(x)](y)$ has dimension $2n$. If $n \neq m$ are both bigger than 1 then $M_n \ncong M_m$ since $M_n(x) \ncong M_m(x)$.

Note that the construction at the end of last paragraph can be generalized to a class of EI-categories which contains a convex subcategory consisting of two non-isomorphic objects x and y such that $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely and transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$. This kind of categories are of infinitely representation type.

As we mentioned in last section, the Eckmann-Shapiro type lemma has a better form when $\mathcal{D} \subset \mathcal{C}$ is full.

Lemma 4.4.21. *Let M be an $R\mathcal{C}$-module which is $\mathcal{D}$-projective for a full subcategory* *of $\mathcal{C}$. If $\mathcal{D}$ is R-co-taut in $\mathcal{C}$, then*

$$\mathrm{Ext}^*_{R\mathcal{C}}(M,N) \cong \mathrm{Ext}^*_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, N\downarrow^{\mathcal{C}}_{\mathcal{D}}).$$

In particular we get $\mathrm{Ext}^*_{R\mathcal{C}}(M,N) \cong \mathrm{Ext}^*_{R\mathcal{C}_M}(M\downarrow^{\mathcal{C}}_{\mathcal{C}_M}, N\downarrow^{\mathcal{C}}_{\mathcal{C}_M})$. *If $M = \underline{R}$ is $\mathcal{D}$-projective and $\mathcal{D}$ is full and R-co-taut in $\mathcal{C}$ then for any $R\mathcal{C}$-module N, we have*

$$\varprojlim{}^*_{\mathcal{C}} N \cong \varprojlim{}^*_{\mathcal{D}} N\downarrow^{\mathcal{C}}_{\mathcal{D}}.$$

Hence if $\mathcal{V}_{\underline{R}}$ is R-co-taut in $\mathcal{C}$ then $\varprojlim^*_{\mathcal{C}} N \cong \varprojlim^*_{\mathcal{V}_{\underline{R}}} N\downarrow_{\mathcal{V}_{\underline{R}}}$ for any $R\mathcal{C}$-module N.

Proof. We can replace the M in Lemma 3.3.6 by $M\downarrow^{\mathcal{C}}_{\mathcal{D}}$, and then use the fact that $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M$. The special cases for higher limits come from the fact that $\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}} = \underline{R}$. □

Note that an R-co-taut $\mathcal{V}_{\underline{R}}$ is in general a rather big subcategory in $\mathcal{C}$, because it contains all minimal objects, and if furthermore it's a co-ideal then we must have $\mathcal{V}_{\underline{R}} = \mathcal{C}$. But in some special cases we may have $\mathcal{V}_{\underline{R}}$ really small. See our further investigations of $\mathcal{V}_{\underline{R}}$ in the next section.

Corollary 4.4.22. *Suppose $\mathcal{C}$ has a unique isomorphism class of minimal objects $[x]$ which forms an R-co-taut full subcategory $\{[x]\}$ in $\mathcal{C}$. If $\{[x]\}$ equals $\mathcal{V}_{\underline{R}}$, then we have $\varprojlim^n_{\mathcal{C}} M \cong \mathrm{Ext}^n_{R\,\mathrm{Aut}(x)}(R, M(x))$ for any $R\mathcal{C}$-module M.*

When we go to the section on categories with subobjects, we will see that any such category with a unique minimal object x will have $\{x\} = \mathcal{V}_{\underline{R}}$, which is indeed R-co-taut in $\mathcal{C}$, and hence we can apply the above result. This can be thought as a generalization of the result for a poset with an initial object (Example 4.3.11). We conclude the applications of the theory of vertices and sources with an isomorphism of cohomology rings.

Proposition 4.4.23. *Let $\mathcal{D} \subset \mathcal{C}$ be a full subcategory, relative to which M is projective. Suppose R is a field or a complete discrete valuation ring and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module. Then there is a ring isomorphism between* $\mathrm{Ext}^*_{R\mathcal{C}}(M,M)$ *and* $\mathrm{Ext}^*_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}})$.

Proof. Let $0 \to M\downarrow^{\mathcal{C}}_{\mathcal{D}} \to N_{n-1} \to \cdots \to N_0 \to M\downarrow^{\mathcal{C}}_{\mathcal{D}} \to 0$ represent an element of $\mathrm{Ext}^n_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}})$ for some positive integer n. Then it gives rise to an element of

$\mathrm{Ext}^n_{R\mathcal{C}}(M, M)$ vis induction $0 \to M \to N_{n-1}\uparrow^{\mathcal{C}}_{\mathcal{D}} \to \cdots \to N_0\uparrow^{\mathcal{C}}_{\mathcal{D}} \to M \to 0$, since $M\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong M$ and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module. But this element of $\mathrm{Ext}^n_{R\mathcal{C}}(M, M)$ restricts back to the given element of $\mathrm{Ext}^n_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}})$. Hence the composite of these two maps $\mathrm{Ext}^n_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}}) \to \mathrm{Ext}^n_{R\mathcal{C}}(M, M) \to \mathrm{Ext}^n_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}})$ is the identity, which implies the first map $\mathrm{Ext}^n_{R\mathcal{D}}(M\downarrow^{\mathcal{C}}_{\mathcal{D}}, M\downarrow^{\mathcal{C}}_{\mathcal{D}}) \to \mathrm{Ext}^n_{R\mathcal{C}}(M, M)$ is injective. Since we know these two Ext groups are isomorphic (Lemma 4.4.21), this map has to be bijective. Now it's easy to check that this map also respects the Yoneda splice, and it defines a ring isomorphism. □

4.5 Structure of the vertex of the trivial module

Because of the special interests in the trivial module $\underline{R}$, we try to obtain a precise description of $\mathcal{D}$, relative to which $\underline{R}$ is projective. Suppose $\underline{R}$ is $\mathcal{D}$-projective and $\mathcal{D}$ is full. Based on Lemma 3.3.3 (2) and the fact $\underline{R} \cong \underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$, we're able to give a simple topological characterization of $\mathcal{D}$ (Proposition 4.5.1). However, this characterization is not easy to use as a criterion to separate such categories $\mathcal{D}$ from others, because it requires understanding of the structures of the overcategories associated to the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$. So we turn to characterize these $\mathcal{D}$ by the "weakly essential objects" which have been introduced and considered in the case of G-posets by authors such as Quillen [34], Puig [32] and Bouc [5]. These weakly essential objects are easy to find and we prove $\mathcal{D}$ must contain them if $\underline{R}$ is $\mathcal{D}$-projective.

Recall that Lemma 3.3.3 (2) says if all overcategories associated to the functor $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ are connected, then $\underline{R}$ is $\mathcal{D}$-projective. Our first proposition will ensure us the converse of that is also true if $\mathcal{D}$ is a full subcategory.

Proposition 4.5.1. *Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be the inclusion of a full subcategory. Then $\underline{R}$ is $\mathcal{D}$-projective if and only if $\iota\downarrow_y$ is connected for any $y \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. By Lemma 3.3.3 (2), we know if $\iota\downarrow_y$ is connected for any $y \in \mathrm{Ob}\,\mathcal{C}$ then $\underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}} \cong K(\underline{R}) \cong \underline{R}$.

On the other hand, if $\underline{R}$ is $\mathcal{D}$-projective we must have $\underline{R} \cong \underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ because $\mathcal{D}$ is a full subcategory. From $K(\underline{R}) \cong \underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$ we get $K(\underline{R})(y) = \varinjlim_{\iota\downarrow_y} \underline{R} \cong \underline{R}(y) = R$. Hence $\iota\downarrow_y$ has to be connected. □

The proposition is not true for subcategories which are not full in $\mathcal{C}$. A simple example will be a group G with a proper Sylow-p subgroup P, both of which are regraded as categories with a single object $*$. When $R = \mathbb{F}_p$, $\underline{R}$ is $\hat{P}$-projective, while $\hat{P} \downarrow_*$ has $[G : P]$ connected components hence is not connected. When $\mathcal{C}$ is a poset, the overcategory $\iota \downarrow_y$ can be identified with $\mathcal{D}_{\leq y}$ for any $y \in \operatorname{Ob}\mathcal{C}$. When $y \in \operatorname{Ob}\mathcal{D}$, $\mathcal{D}_{\leq y}$ has a terminal object and thus is contractible. When $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, we have $\mathcal{D}_{\leq y} = \mathcal{D}_{<y}$, and the connectedness of the latter has been studied using the so-called "weakly essential objects" by Puig [32], Quillen [34] and Bouc [5]. In fact, the weakly essential objects are quite useful in analyzing the topological structures of subposets of $\mathcal{C}$ which contain them. Because of this and that every EI-category has an underlying poset, we would like to see if they can be used in our case. It turns out the answer is positive.

Recall that for an EI-category $\mathcal{C}$ and an object $x \in \operatorname{Ob}\mathcal{C}$, we denote by $\mathcal{C}_{<x}$ the full subcategory whose objects are $\{y \in \operatorname{Ob}\mathcal{C} | \operatorname{Hom}_{\mathcal{C}}(y, x) \neq \emptyset, y \ncong x\}$.

Definition 4.5.2. *An object $x \in \operatorname{Ob}\mathcal{C}$ is weakly essential if the full subcategory $\mathcal{C}_{<x}$ is empty or has more than one component. The full subcategory of $\mathcal{C}$, consisting of all weakly essential objects, is named $\operatorname{Wess}_0(\mathcal{C})$. There is a larger full subcategory $\operatorname{Wess}(\mathcal{C})$ consists of objects $x \in \operatorname{Ob}\mathcal{C}$ which satisfy the condition that the full subcategory $\mathcal{C}_{<x}$ are not contractible.*

Obviously the minimal objects of $\mathcal{C}$ are weakly essential, contained in $\operatorname{Wess}_0(\mathcal{C})$. The notations $\operatorname{Wess}_0(\mathcal{C})$ and $\operatorname{Wess}(\mathcal{C})$ belong to Symonds [37], though the first was studied by Puig [32] and the second by Quillen [34] and Bouc (see [4] Definition 6.6.2). We've seen in last section that all minimal objects are contained in $\mathcal{V}_{\underline{R}}$. This will be strengthened shortly by showing $\operatorname{Wess}_0(\mathcal{C}) \subset \mathcal{V}_{\underline{R}}$. Furthermore when $\mathcal{C}$ is a poset, $\mathcal{V}_{\underline{R}}$ is totally determined by $\operatorname{Wess}_0(\mathcal{C})$.

Let $\underline{R} \cong \underline{R}\downarrow^{\mathcal{C}}_{\mathcal{D}}\uparrow^{\mathcal{C}}_{\mathcal{D}}$. Since $\underline{R}$ is $\mathcal{D}$-projective, it is generated by its values on $\mathcal{D}$. If we take an object $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, then $\underline{R}(y)$ is generated by $\underline{R}$'s values on $\mathcal{D}_{<y}$. This forces $\mathcal{D}_{<y}$ to be connected, because otherwise $\underline{R}(y)$ would have an R-rank bigger than one which is impossible.

Lemma 4.5.3. *Let $\mathcal{D}$ be a connected full subcategory of $\mathcal{C}$. Suppose $\underline{R}$ is $\mathcal{D}$-projective. Then for any $y \notin \operatorname{Ob}\mathcal{D}$, $\mathcal{D}_{<y}$ is non-empty and connected.*

Proof. Obviously $\mathcal{D}_{<y}$ is non-empty, because otherwise it'll contradict with the definition of $\mathcal{D}$. If $\mathcal{D}_{<y}$ were disconnected, then we prove

$$\underline{R}\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = \sum_{x \in \operatorname{Ob}\mathcal{D}} R\operatorname{Hom}(x,y) \otimes \underline{R}(x)$$

is a direct sum of at least two non-zero summands. Hence a contradiction since $\underline{R}\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = R$.

If $\mathcal{D}_{<y}$ were disconnected, then $\underline{R}\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = \sum_{x\in\operatorname{Ob}\mathcal{D}} R\operatorname{Hom}(x,y)\otimes\underline{R}(x)$ contains two elements $\alpha\otimes 1$ and $\beta\otimes 1$, where $\alpha\in\operatorname{Hom}(x_1,y)$ and $\beta\in\operatorname{Hom}(x_2,y)$ for x_1, x_2 from different components of $\mathcal{D}_{<y}$. Let's assume x_1, x_2 minimal. Now, since $\underline{R}\downarrow_{\mathcal{D}}^{\mathcal{C}}\uparrow_{\mathcal{D}}^{\mathcal{C}}(y) = R$ has rank 1, we have $r\alpha\otimes_{R\mathcal{D}}1 = \beta\otimes_{R\mathcal{D}}1$ for some $r\in R$. But it means that $r\alpha\gamma\otimes_{R\mathcal{D}}1_{x_2} = \beta\otimes_{R\mathcal{D}}1_{x_2}$ (or $r\alpha\otimes_{R\mathcal{D}}1_{x_1} = \beta\gamma\otimes_{R\mathcal{D}}1_{x_1}$) for some $\gamma\in R\operatorname{Hom}(x_2,x_1)$ (or in $R\operatorname{Hom}(x_1,x_2)$), which implies $\operatorname{Hom}(x_2,x_1)$ (or $\operatorname{Hom}(x_1,x_2)$) is non-empty. So x_1 and x_2 belong to the same connected component which is a contradiction. □

The above result leads us to a corollary which is more convenient to use than Proposition 4.5.1 as a criterion for the $\mathcal{D}$-projectivity of $\underline{R}$.

Corollary 4.5.4. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, relative to which $\underline{R}$ is projective. Then* $\operatorname{Wess}_0(\mathcal{C})\subset\mathcal{D}$. *In particular* $\operatorname{Wess}_0(\mathcal{C})\subset\mathcal{V}_{\underline{R}}$.

Proof. We show if $y\notin\operatorname{Ob}\mathcal{D}$, then $y\notin\operatorname{Wess}_0(\mathcal{C})$. First of all since $\mathcal{D}$ contains all minimal objects in $\mathcal{C}$, $\mathcal{C}_{<y}$ can not be empty because $y\notin\operatorname{Ob}\mathcal{D}$. We claim $\mathcal{C}_{<y}$ is connected. Assume the opposite. Since $\mathcal{D}_{<y}\subset\mathcal{C}_{<y}$ and $\mathcal{C}_{<y}$ is disconnected, by Lemma 4.5.2 $\mathcal{D}_{<y}$ must lie in only one of the components of $\mathcal{C}_{<y}$. But then $\mathcal{C}_{<y}$ contains at least one minimal object x which doesn't belong to $\operatorname{Ob}\mathcal{D}_{<y}$. Actually, x isn't in $\operatorname{Ob}\mathcal{D}$ either, because otherwise $\mathcal{D}_{<y}$ is disconnected. Now $x\notin\operatorname{Ob}\mathcal{D}$ contradicts with the fact that $\mathcal{D}$ contains all minimal objects. □

The following result extends a theorem of Bouc (Proposition 4.6.7) on posets, saying that $\operatorname{Wess}_0(\mathcal{C})$ and $\mathcal{C}$ have the same numbers of connected components. Recall that there is a poset $P(\mathcal{C})$ associated to every EI-category $\mathcal{C}$. When $\mathcal{C}$ is finite we call the maximal length of chains of non-isomorphisms in $P(\mathcal{C})$ the length of $\mathcal{C}$, denoted by $l(\mathcal{C})$.

Lemma 4.5.5. *Let $\mathcal{D}\subset\mathcal{C}$ be a full subcategory. If* $\operatorname{Wess}_0(\mathcal{C})\subset\mathcal{D}$, *then* $\operatorname{Wess}_0(\mathcal{C})\subset\mathcal{D}\subset\mathcal{C}$ *induce bijections on connected components.*

Proof. Firstly we assume $\mathcal{C}$ is connected and prove $\mathcal{D}$ has to be connected too. We do induction on the length of $\mathcal{C}$. Assume $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Then for any $y \in \mathrm{Ob}\,\mathcal{C}$, from $\mathrm{Wess}_0(\mathcal{C})_{<y} = \mathrm{Wess}_0(\mathcal{C}_{<y})$ and $\mathrm{Wess}_0(\mathcal{C}_{<y}) \subset \mathcal{D}_{<y} \subset \mathcal{C}_{<y}$ we deduce that $\mathcal{D}_{<y} = \mathcal{D}_{1<y} \cup \mathcal{D}_{2<y}$ is connected since $l(\mathcal{C}_{<y}) < l(\mathcal{C})$ and $\mathcal{C}_{<y}$ is connected. Thus for every non-minimal $y \in \mathrm{Ob}\,\mathcal{C}$, one of the $\mathcal{D}_{i<y}$, $i = 1, 2$ must be empty. Let Y_1 be the set of maximal objects in $\mathcal{C}$ such that $\mathcal{D}_{2<y} = \emptyset$ and Y_2 be the set of maximal objects such that $\mathcal{D}_{1<y} = \emptyset$. If both Y_1 and Y_2 were not empty, then for $y_1 \in Y_1$ and $y_2 \in Y_2$ we must have $\mathcal{C}_{<y_1} \cap \mathcal{C}_{<y_2} = \emptyset$ because otherwise it would contain a minimal object z which belongs to $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}_1 \cup \mathcal{D}_2$. Hence $z \in \mathcal{D}_{1<y_2}$ (or $z \in \mathcal{D}_{2<y_1}$), a contradiction to the definition of Y_1 and Y_2. But $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$ leads to an equality $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_i = \{x \in \mathrm{Ob}\,\mathcal{C} \mid x \in \mathrm{Ob}\,\mathcal{C}_{<y}, y \in Y_i\}$, $i = 1, 2$. This is impossible. Thus either Y_1 or Y_2 is empty, which means either $\mathcal{D}_1$ or $\mathcal{D}_2$ is empty. Hence $\mathcal{D}$ is connected.

Secondly suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_n$, where $\mathcal{C}_i$, $i = 1, \cdots, n$ are connected components. We show $\mathcal{D}$ has the same number of components. Let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \cdots \cup \mathcal{D}_m$ as a union of connected components. Then every $\mathcal{D}_j$ is contained in some $\mathcal{C}_i$, while no two distinct $\mathcal{D}_j$'s are contained in the same $\mathcal{C}_i$ because of the above arguments and the fact that $\mathrm{Wess}_0(\mathcal{C}) = \mathrm{Wess}_0(\mathcal{C}_1) \cup \cdots \cup \mathrm{Wess}_0(\mathcal{C}_n)$. This forces $m = n$ and induces a natural bijection on the connected components of $\mathcal{D}$ and $\mathcal{C}$. We're done. □

If $\mathcal{C}$ is connected and $\underline{R}$ is $\mathcal{D}$-projective, then $\mathcal{D}$ is connected because we always have $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$.

Corollary 4.5.6. *Let $\mathcal{C}$ be an EI-category and $\mathcal{D}$ a connected full subcategory, relative to which $\underline{R}$ is projective. For every $y \notin \mathrm{Wess}_0(\mathcal{C})$, $\mathrm{Wess}_0(\mathcal{C})_{<y}, \mathrm{Wess}(\mathcal{C})_{<y}$ and $\mathcal{D}_{<y}$ are all connected.*

Proof. By definition of $\mathrm{Wess}_0(\mathcal{C})$, $\mathcal{C}_{<y}$ is always connected. The results follow from the inclusions $\mathrm{Wess}_0(\mathcal{C})_{<y} \subset \mathcal{D}_{<y} \subset \mathcal{C}_{<y}$ and $\mathrm{Wess}_0(\mathcal{C})_{<y} \subset \mathrm{Wess}(\mathcal{C})_{<y} \subset \mathcal{C}_{<y}$, combined with Lemma 4.5.5 and the fact that $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$. □

When $\mathcal{C}$ is a poset and $\mathcal{D}$ is a subposet containing $\mathrm{Wess}_0(\mathcal{C})$, by $\iota \downarrow_y \cong \mathcal{D}_{\leq y}$ and Proposition 4.5.1, we see $\underline{R}$ is $\mathcal{D}$-projective, because if $y \in \mathrm{Ob}\,\mathcal{D}$ then $\iota \downarrow_y$ is contractible and if $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$, $\mathrm{Wess}_0(\mathcal{C}_{<y}) \subset \mathcal{D}_{<y} \subset \mathcal{C}_{<y}$ implies $\mathcal{D}_{<y}$ hence $\iota \downarrow_y$ is connected. Thus as a special case $\underline{R}$ is $\mathrm{Wess}_0(\mathcal{C})$-projective and $\mathrm{Wess}_0(\mathcal{C})$

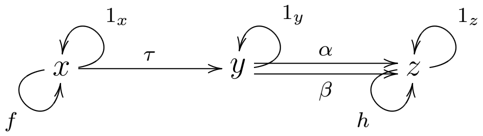
becomes the smallest full category, relative to which $\underline{R}$ is projective. This means $\mathcal{V}_{\underline{R}}$ is the "convex hull" of $\mathrm{Wess}_0(\mathcal{C})$, which is uniquely determined by $\mathrm{Wess}_0(\mathcal{C})$. We give a sufficient condition on the connectedness of overcategories and the relative projectivity of $\underline{R}$.

Proposition 4.5.7. *Let $\mathcal{C}$ be an EI-category and $\mathcal{D}$ a full subcategory containing $\mathrm{Wess}_0(\mathcal{C})$. Then every $\iota \downarrow_y$, $y \in \mathrm{Ob}\,\mathcal{C}$, is connected if for any pair of objects $x \in \mathrm{Ob}\,\mathcal{D}$ and $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$, $\mathrm{Aut}_{\mathcal{C}}(x)$ acts transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$. Hence $\underline{R}$ is $\mathcal{D}$-projective.*

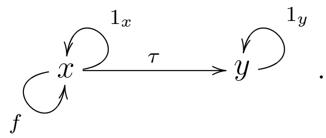
Proof. According to Lemma 3.3.3 (2) and Proposition 4.5.1 we only need to show the overcategory $\iota \downarrow_y$ is connected for each $y \in \mathrm{Ob}\,\mathcal{C}$. When $y \in \mathrm{Ob}\,\mathcal{D}$, $\iota \downarrow_y$ is connected because there is only one isomorphism class of maximal objects, of the form (y, g) where $g \in \mathrm{Aut}(y)$. Now we assume $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$. The objects of $\iota \downarrow_y$ are of the form (x, α), where $x \in \mathrm{Ob}\,\mathcal{D}_{<y}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. If for every pair of objects $x \in \mathrm{Ob}\,\mathcal{D}$ and $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$, $\mathrm{Aut}_{\mathcal{C}}(x)$ acts transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$, then $\iota \downarrow_y$ has the same underlying poset as $\mathcal{D}_{<y}$ because $(x, \alpha) \cong (x, \beta)$ for any two morphisms $\alpha, \beta \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. Now $\mathcal{D}_{<y}$ is connected if and only if its underlying poset is connected. Since $\mathrm{Wess}_0(\mathcal{C}) \subset \mathcal{D}$, we know $\mathcal{D}_{<y}$ is connected for any $y \in \mathrm{Ob}\,\mathcal{C}$ by Corollary 4.5.6. $\square$

In some sense, the above lemma tells us what objects in $\mathcal{C}$ are needed to be kept in $\mathcal{D}$ in order to make $\underline{R}$ $\mathcal{D}$-projective, which could be used to find $\mathcal{V}_{\underline{R}}$. For instance in Example 4.5.8 below, we have a category with three objects: x, y and z. Since the automorphism groups $\mathrm{Aut}_{\mathcal{C}}(x)$ and $\mathrm{Aut}_{\mathcal{C}}(y)$ do not act transitively on the corresponding sets of non-isomorphisms ending at z, z is contained in $\mathcal{V}_{\underline{R}}$. Furthermore since $\mathrm{Wess}_0(\mathcal{C}) = \{x\}$, $\mathcal{V}_{\underline{R}}$ must contain x and z. In fact the smallest full subcategory $\mathcal{D} \subset \mathcal{C}$, relative to which $\underline{R}$ is projective, consisting of exactly the two objects x and z, while $\mathcal{V}_{\underline{R}} = \mathcal{C}$ because it needs to be convex. Note that in the example, we also show that in general there is no containment between $\mathrm{Wess}(\mathcal{C})$ and $\mathcal{D}$, even if $\underline{R}$ is $\mathcal{D}$-projective.

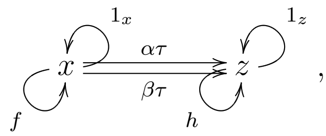
Example 4.5.8. *Suppose $\mathcal{C}$ is the following category*



with R arbitrary, $\mathrm{Aut}_{\mathcal{C}}(x)$ acting trivially on $\mathrm{Hom}_{\mathcal{C}}(x,y) = \{\tau\}$, $\mathrm{Aut}_{\mathcal{C}}(z)$ interchanging α and β, and $\beta\tau \neq \alpha\tau$. From direct calculations, we can see that $\mathcal{V}_{\underline{R}} = \mathcal{C}$. The following category is $\mathrm{Wess}(\mathcal{C})$



We can check $\underline{R}$ is projective relative to the full subcategory $\mathcal{D}$



which is not convex and is the smallest full subcategory among all those relative to which $\underline{R}$ is projective. Comparing these categories, we get $\mathrm{Wess}(\mathcal{C}) \subset \mathcal{V}_{\underline{R}} = \mathcal{C}$, $\mathcal{D} \not\subset \mathrm{Wess}(\mathcal{C})$ *and* $\mathrm{Wess}(\mathcal{C}) \not\subset \mathcal{D}$. *Note that* $\mathrm{Wess}_0(\mathcal{C}) = \{x\}$ *is contained in* $\mathcal{V}_{\underline{R}}$, $\mathrm{Wess}(\mathcal{C})$ *and $\mathcal{D}$.*

We end this section with a little bit further discussion of the structure of the overcategories. The first result describes the maximal objects in the overcategories when $\mathcal{D} \subset \mathcal{C}$ is full. It somehow reveals certain connections between representation-theoretic and topological aspects of the overcategories, because we will see the maximal objects are related to the irreducible morphisms. The second and the third results continue the discussion for $\mathcal{D} \subset \mathcal{C}$ that is not full.

Lemma 4.5.9. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ along with the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$. Then*

1. *if $y \in \mathrm{Ob}\,\mathcal{D}$, an object $(x,\alpha) \in \mathrm{Ob}\,\iota\uparrow^y$ is maximal if and only if $x \cong y$ and $\alpha \in \mathrm{Is}_{\mathcal{C}}(x,y)$; and*

2. *if $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$, an object $(x,\alpha) \in \mathrm{Ob}\,\iota\uparrow^y$ is maximal if and only if $\alpha \in \mathrm{coIrr}_{\mathcal{D}}(x,y)$.*

Proof. Just use the definitions of $\iota\downarrow_y$ and $\mathrm{coIrr}_{\mathcal{D}}(x,y)$. $\square$

This result can be used to establish Proposition 4.5.1. When $y \in \mathrm{Ob}\,\mathcal{D}$, we know that $\iota\downarrow_y$ is always contractible, since its skeleton has a terminal object, i.e. the isomorphism class of $(y, 1_y)$. If $\underline{R}$ is $\mathcal{D}$-projective, then for every $y \in \mathrm{Ob}\,\mathcal{C} \setminus \mathrm{Ob}\,\mathcal{D}$ we have $(R\mathcal{C} \otimes_{R\mathcal{D}} \underline{R})(y) = \sum_{y>x\in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{Hom}(x,y) \otimes_{R\mathcal{D}} 1_x = R$. Furthermore we can write $\sum_{y>x\in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{coIrr}_{\mathcal{D}}(x,y) \otimes_{R\mathcal{D}} 1_x = R$. Now by Lemma 4.5.9, $(x,\alpha) \in \iota\downarrow_y$, for $\alpha \in \mathrm{coIrr}_{\mathcal{D}}(x,y)$, is a maximal object, and indeed all maximal objects are of this form. If $\iota\downarrow_y$ were disconnected, then there are two maximal objects, say (x,α) and (z,β), belonging to two different components of $\iota\downarrow_y$, which implies that $\alpha \otimes_{R\mathcal{D}} 1_x$ and $\beta \otimes_{R\mathcal{D}} 1_z$ are linearly independent. Hence $\sum_{y>x\in \mathrm{Ob}\,\mathcal{D}} R\,\mathrm{coIrr}_{\mathcal{D}}(x,y) \otimes_{R\mathcal{D}} 1_x$ has rank > 1, a contradiction.

If $\mathcal{D} \subset \mathcal{C}$ is a full subcategory, ι is the inclusion and $y \in \mathrm{Ob}\,\mathcal{D}$, then the skeleton of $\iota\downarrow_y$ has a terminal object given by $(y, 1_y)$, hence $\iota\downarrow_y$ is connected (contractible indeed). When $\mathcal{D}$ is not full in $\mathcal{C}$, $\iota\downarrow_y$ is not necessarily connected, and there usually exists more than one maximal object in each connected component. We count the number of isomorphism classes of maximal objects in $\iota\downarrow_y$ if $y \in \mathrm{Ob}\,\mathcal{D}$.

Lemma 4.5.10. *Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ along with the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$. Suppose $y \in \mathrm{Ob}\,\mathcal{D}$. Then the number of isomorphism classes of maximal objects of $\iota\downarrow_y$ is exactly $|\mathrm{Aut}_{\mathcal{C}}(y) : \mathrm{Aut}_{\mathcal{D}}(y)|$. Furthermore the isomorphism class of any (y,g), $g \in \mathrm{Aut}_{\mathcal{C}}(y)$, contains exactly $|\mathrm{Aut}_{\mathcal{D}}(y)|$ maximal objects, and each maximal object has a trivial automorphism group.*

Proof. Without loss of generality, let's assume $\mathcal{C}$ is skeletal. Then $(y,g) \cong (y,g')$ if and only if there are in the same (left) coset of $[\mathrm{Aut}_{\mathcal{C}}(y)/\mathrm{Aut}_{\mathcal{D}}(y)]$. $\square$

These maximal objects may lie in different connected components of $\iota\downarrow_y$. When $\mathcal{D} \subset \mathcal{C}$ isn't full, it's not easy to count the number of the connected components, even if $y \in \mathrm{Ob}\,\mathcal{D}$. A connected component may contain two non-isomorphic maximal objects (y,g) and (y,g') as long as there exists a (x,α) and $\beta, \beta' \in \mathrm{Hom}(x,y)$ such that $\alpha = g\beta$ and $\alpha = g'\beta'$. From $g\beta = g'\beta'$ we get $\beta = g^{-1}g'\beta'$ where $g^{-1}g' \in \mathrm{Aut}_{\mathcal{C}}(y) \setminus \mathrm{Aut}_{\mathcal{D}}(y)$. This proves the following.

Lemma 4.5.11. *Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be an inclusion and $y \in \mathrm{Ob}\,\mathcal{D}$. Then some connected component of $\iota\downarrow_y$ contains more than one isomorphism class of maximal objects if* *and only if there exists an* $x \in \operatorname{Ob}\mathcal{D}_{<y}$ *and a* $\beta \in \operatorname{Hom}(x, y)$*, so that* $|\operatorname{Aut}_{\mathcal{C}}(y) \cdot \beta|$ *is strictly bigger than* $|\operatorname{Aut}_{\mathcal{D}}(y) \cdot \beta|$.

It's easy to see that $\operatorname{Aut}_{\mathcal{C}}(y)$ acts on the set of connected components of $\iota \downarrow_y$ via $g : (x, \alpha) \mapsto (x, g\alpha)$, because it permutes the set of maximal objects. When $y \in \operatorname{Ob}\mathcal{D}$, $\operatorname{Aut}_{\mathcal{C}}(y)$ acts transitively and the stabilizer of this action contains $\operatorname{Aut}_{\mathcal{D}}(y)$. When $y \notin \operatorname{Ob}\mathcal{D}$, the action of $\operatorname{Aut}_{\mathcal{C}}(y)$ is not transitively in general, since if there are two components then $\operatorname{Aut}_{\mathcal{C}}(y)$ cannot send a maximal object in one component to one in another. This is so because the maximal objects in $\iota \downarrow_y$ are those objects (x, α) with $\alpha \in \operatorname{coIrr}_{\mathcal{D}}(x, y)$ and $\operatorname{Aut}_{\mathcal{C}}(y) \operatorname{coIrr}_{\mathcal{D}}(x, y) = \operatorname{coIrr}_{\mathcal{D}}(x, y)$. It's not clear how to tell whether or not $\iota \downarrow_y$ is connected, when $y \notin \operatorname{Ob}\mathcal{D}$. But obviously when $x \not\cong x'$, $(x, \alpha) \not\cong (x', \alpha')$, and when $x \cong x'$, $(x, \alpha) \cong (x', \alpha')$ if and only if there is an $f \in \operatorname{Is}_{\mathcal{D}}(x, x')$ such that $f\alpha' = \alpha$.

4.6 Categories with subobjects

This part of the work grows out of our observation that if $\mathcal{C}$ is a finite poset then $\mathcal{V}_{\underline{R}}$ is determined by $\operatorname{Wess}_0(\mathcal{C})$, see comments after Corollary 4.5.6. Let $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ be the inclusion of a connected full subcategory. Proposition 4.5.1 tells us that the $R\mathcal{C}$-module $\underline{R}$ is $\mathcal{D}$-projective if and only if every overcategory $\iota \downarrow_y$, $y \in \operatorname{Ob}\mathcal{C}$, is connected. We mentioned before that when $\mathcal{C}$ is a poset, the overcategory $\iota \downarrow_y$ can be identified with a subposet $\mathcal{D}_{\leq y}$ for each $y \in \operatorname{Ob}\mathcal{C}$. Since the connectedness and the contractibility of such subposets determine useful topological properties of the poset $\mathcal{C}$ through $\operatorname{Wess}_0(\mathcal{C})$ and $\operatorname{Wess}(\mathcal{C})$, we cannot help wondering if there are other categories such that the overcategories associated to the inclusion of full subcategories are isomorphic to posets, and to what extent these posets are related to the weakly essential objects? In this section, our focus goes to the categories with subobjects, which were introduced and studied by Oliver [31] and include posets as special cases. A category with subobjects is usually dubbed as $(\mathcal{C}, \mathcal{I})$, where $\mathcal{C}$ is a small category and $\mathcal{I}$ is a subcategory (see definition below). We show any finite category with subobjects $(\mathcal{C}, \mathcal{I})$ is EI, and $\mathcal{I}$ is indeed a poset. One of the key properties of an EI-category with subobjects is that all its morphisms are monomorphic. Given $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ we deduce that the overcategories in question can be identified with subposets of $\mathcal{I}$.

This means some homological properties of EI-categories with subobjects are governed by the subposets of $\mathcal{I}$ so that we can extend a Bouc's result to these categories. Investigation in this direction goes beyond what prompts it, and we've recorded some properties of EI-categories with subobjects which are not necessarily related to our studies in the preceding sections. But as an explicit and generalized proposition of our opening statement, we can say, given a finite category with subobjects $(\mathcal{C}, \mathcal{I})$, $\mathrm{Wess}_0(\mathcal{I})$ uniquely determines the smallest full subcategory, relative to which $\underline{R}$ is projective. In fact $\mathcal{V}_{\underline{R}}$ can be described as the "full convex hull" of $\mathrm{Wess}_0(\mathcal{I})$.

Following Oliver's original definition, *a category with subobjects* is a pair of categories $\mathcal{I} \subset \mathcal{C}$ such that $\mathrm{Ob}\,\mathcal{I} = \mathrm{Ob}\,\mathcal{C}$, and such that the following two conditions are satisfied:

1. $|\mathrm{Hom}_{\mathcal{I}}(x,y)| \leq 1$ for any pair of objects x, y; and

2. each morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ can be written in a unique way as a composite $\alpha = \alpha_0 \cdot f$, where $f \in \mathrm{Is}_{\mathcal{C}}(x,x')$ for some x', and $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x',y)$.

When $\mathcal{C}$ is EI, the morphisms in $\mathrm{Mor}(\mathcal{I})$ are necessarily monomorphic. From a category-theoretic viewpoint, if there is a monomorphism from x to y, two objects in some category $\mathcal{C}$, then x is called a subobject of y. If furthermore the category $\mathcal{C}$ is concrete, i.e. every object has an underlying set structure, then the meaning of the term "subobject" is obvious.

Example 4.6.1. *Let mSets be the category of finite sets, whose morphisms are all monomorphic functions. It's natural to choose $\mathcal{I}_{mSets}$ to be the (connected) subcategory whose morphisms are inclusions, which is indeed a connected poset since mSets is an EI-category. But it's easy to see there are other choices of $\mathcal{I}$. In $(mSets, \mathcal{I}_{mSets})$, we fix two sets A and B, of sizes i and j, respectively, where i, j are two positive integers with $i \leq j$. Then any $\alpha \in \mathrm{Hom}_{mSets}(A,B)$ can be uniquely factored as $\alpha_0 f$, where $f \in \mathrm{Is}_{mSets}(A, \alpha(A))$ and $\alpha_0 \in \mathrm{Hom}_{mSets}(\alpha(A), B)$ is the inclusion (in $\mathcal{I}_{mSets}$). Note that $|\mathrm{Hom}_{mSets}(A,B)| = C^i_j \cdot i!$ (here C^i_j means j choose i), and $|\mathrm{End}_{mSets}(A)| = |\mathrm{Aut}_{mSets}(A)| = |S_i| = i!$. A surprising fact is that $\mathrm{Hom}_{\mathcal{I}_{mSets}}(x,y)$ can be empty even if $\mathrm{Hom}_{mSets}(x,y)$ is not. Consider two sets A and B, and suppose $\mathrm{Hom}_{mSets}(A,B)$ is non-empty. Then as objects in $\mathcal{I}_{mSets}$, $\mathrm{Hom}_{\mathcal{I}_{mSets}}(A,B) \neq \emptyset$ if and only if A is a subset of B. Particularly two objects do not have to be isomorphic in $\mathcal{I}_{mSets}$, even* *if they are isomorphic in mSets, since two finite sets of the same cardinality are isomorphic in mSets, but not in $\mathcal{I}_{mSets}$ unless they are equal. Because mSets is an EI-category, there is no non-isomorphism between two isomorphic objects. Hence the picture of $\mathcal{I}_{mSets}$ is very clear.*

In this example $\mathcal{I}_{mSets}$ is connected, which is not necessarily true as a general rule. For instance we can consider the (infinite) full subcategory $mSets_{\leq B}$. There is a (finite) full subcategory $mSets_{\subseteq B} \subset mSets_{\leq B}$, whose objects are the subsets of B. These two categories are categories with subobjects with distinct $\mathcal{I}$'s. If we name the subcategories as $\mathcal{I}_{\subseteq B}$ and $\mathcal{I}_{\leq B}$, then $\mathcal{I}_{\leq B}$ is the disjoint union of $\mathcal{I}_{\subset B}$ and infinitely many discrete objects, which are isomorphic, but not equal, to an object in $mSets_{\subseteq B}$.

We say $(\mathcal{C}, \mathcal{I})$ is a skeletal category with subobjects, if $\mathcal{C}$ is skeletal (this implies $\mathcal{I}$ is skeletal as well). It is *not* true that if $(\mathcal{C}, \mathcal{I})$ is a category with subobjects, then the skeleton of $\mathcal{C}$ can be made into a category with subobjects. A convenient example will be the category *mSets* we just mentioned. The skeleton of *mSets*, $\overline{mSets}$, can be constructed by taking one set from each isomorphism class and then form a full subcategory with these sets. Hence its object set consists of finite sets of pairwise distinct sizes. This category cannot be made into a category with subobjects since $\mathrm{End}_{mSets}(A)$ acts freely, but not transitively, on $\mathrm{Hom}_{\overline{mSets}}(A, B) = \mathrm{Hom}_{mSets}(A, B)$ whenever $|A| < |B|$ (hence there is no way to find a suitable $\mathcal{I}_{\overline{mSets}}$).

If we assume $\mathcal{C}$ is an EI-category, then we naturally have a definition of EI-categories with subobjects. When $(\mathcal{C}, \mathcal{I})$ is an EI-category with subobjects, the subcategory $\mathcal{I}$ becomes a poset (Lemma 4.6.2) whose structure is usually different from the underlying poset $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$ (one can check this for the category *mSets*). We note that Jackowski and Słomińska introduced EI-categories with quotients in their paper [25], which is the dual concept to EI-categories with subobjects. All results in this section have their counterparts for EI-categories with quotients, because the opposite of an EI-category with quotients is an EI-category with subobjects, and vice versa.

Lemma 4.6.2. *Let $(\mathcal{C}, \mathcal{I})$ be a category with subobjects. Then the endomorphism group of each $x \in \mathrm{Ob}\,\mathcal{I}$ is precisely $\{1_x\}$, and the only isomorphisms in $\mathrm{Mor}(\mathcal{I})$ are these 1_x, for all $x \in \mathrm{Ob}\,\mathcal{I}$. If furthermore $\mathcal{C}$ is EI, $\mathcal{I}$ is a poset.*

Proof. It's easy to see $\mathrm{End}_{\mathcal{I}}(x) = \mathrm{Aut}_{\mathcal{I}}(x) = \{1_x\}$ since $|\mathrm{End}_{\mathcal{I}}(x)| \leq 1$ and $1_x \in \mathrm{End}_{\mathcal{I}}(x)$. Now we show $\mathrm{Is}_{\mathcal{I}}(x, y) = \emptyset$ if $x \cong y$ (in $\mathcal{C}$) and $x \neq y$. If there were an

$\alpha_0 \in \mathrm{Is}_{\mathcal{I}}(x,y)$, then we would have two distinct factorizations $\alpha_0 = 1_y\alpha_0 = \alpha_0 1_x$, a contradiction to the definition of a category with subobjects.

If $\mathcal{C}$ is EI and $\mathrm{Hom}_{\mathcal{I}}(x,y) \neq \emptyset$ for some $x \neq y$, we show $\mathrm{Hom}_{\mathcal{I}}(y,x) = \emptyset$. This implies $\mathcal{I}$ is a poset. Indeed if there exists $\alpha \in \mathrm{Hom}_{\mathcal{I}}(x,y)$ and $\beta \in \mathrm{Hom}_{\mathcal{I}}(y,x)$, then $\alpha\beta = 1_y$ and $\beta\alpha = 1_x$ in $\mathcal{I}$ (and $\mathcal{C}$). Hence α and β are isomorphisms, which is impossible because from above we know $\mathrm{Is}_{\mathcal{I}}(x,y) = \emptyset$ if $x \neq y$. □

The above result implies that if $(\mathcal{C},\mathcal{I})$ is a (not necessarily EI) category with subobjects, a non-empty set $\mathrm{Hom}_{\mathcal{I}}(x,y)$ (if there is any) will consist of a non-isomorphism, given that $x \cong y$ and $x \neq y$ in $\mathcal{C}$. This is the situation we want to avoid. Our next result says, if $\mathcal{C}$ satisfies a particular condition, then we don't have to worry about it.

Proposition 4.6.3. *Let $(\mathcal{C},\mathcal{I})$ be a category with subobjects. If every isomorphism class of objects in $\mathcal{C}$ is finite, then $\mathcal{C}$ is an EI-category.*

Proof. Suppose there exists an $\alpha \in \mathrm{End}_{\mathcal{C}}(x)\backslash \mathrm{Aut}_{\mathcal{C}}(x)$ for some $x \in \mathrm{Ob}\,\mathcal{C}$. We show this assumption leads to a contradiction. By definition of a category with subobjects, $\alpha = \alpha_0 f$ for some $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x',x)$ and $f \in \mathrm{Is}_{\mathcal{C}}(x,x')$, where $x' \cong x$ in $\mathcal{C}$. We claim $x' \neq x$. If $x = x'$, then $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x,x) = \mathrm{End}_{\mathcal{I}}(x) = \{1_x\}$ by preceding lemma. But then $\alpha = f$ is an isomorphism, a contradiction to our assumption. From $x \neq x'$, we know $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(x',x)$ is not an isomorphism.

Now since $x' \cong x$, there exists a $\beta \in \mathrm{End}_{\mathcal{C}}(x')\backslash \mathrm{Aut}_{\mathcal{C}}(x')$, and $\beta = \beta_0 g$ for some $\beta_0 \in \mathrm{Hom}_{\mathcal{I}}(x'',x')$ and $g \in \mathrm{Is}_{\mathcal{C}}(x',x'')$. As is shown in last paragraph, $x'' \neq x'$ and β_0 is not an isomorphism. In fact x'' cannot be x either, since if they were equal, we would have two morphisms in $\mathcal{I}$: $\beta_0 : x \to x'$ and $\alpha_0 : x' \to x$, which implies $\alpha_0\beta_0 = 1_x$ and $\beta_0\alpha_0 = 1_{x'}$. The two equalities assert that α_0 and β_0 are isomorphisms, inverse to each other, hence a contradiction to our assumptions on α_0 and β_0. Thus any two of x, x' and x'' are not equal, and we can find a third non-isomorphism $\gamma \in \mathrm{End}_{\mathcal{C}}(x'')\backslash \mathrm{Aut}_{\mathcal{C}}(x'')$ so that we can repeat what we've done for $\beta \in \mathrm{End}_{\mathcal{C}}(x')\backslash \mathrm{Aut}_{\mathcal{C}}(x')$. Gradually, we're going to produce an infinite list of isomorphic objects in $\mathcal{C}$, $x, x', x'', \cdots$, while any two of them are not equal. This leads to a contradiction since we assume $[x]$ is finite. Thus there is no such $\alpha \in \mathrm{End}_{\mathcal{C}}(x)\backslash \mathrm{Aut}_{\mathcal{C}}(x)$ for any $x \in \mathrm{Ob}\,\mathcal{C}$, and then $\mathrm{End}_{\mathcal{C}}(x) = \mathrm{Aut}_{\mathcal{C}}(x)$ for all $x \in \mathrm{Ob}\,\mathcal{C}$, or $\mathcal{C}$ is EI. □

Categories with subobjects constructed from subgroups of (finite) groups have the property that each isomorphism class of objects is finite. Hence they are always EI. More generally, any finite category with subobjects $(\mathcal{C}, \mathcal{I})$ is EI, and $\mathcal{I}$ is a poset. When $(\mathcal{C}, \mathcal{I})$ is skeletal, $\mathcal{C}$ is automatically an EI-category as a corollary of last result. However, we can prove it directly using the definition of a category with subobjects.

Corollary 4.6.4. *Let $(\mathcal{C}, \mathcal{I})$ be a skeletal category with subobjects. Then $\mathcal{C}$ is an EI-category, and $(\mathcal{C}, \mathcal{I})$ is an EI-category with subobjects. In fact $\mathcal{I} \cong P(\mathcal{C})$, the underlying poset of $\mathcal{C}$.*

Proof. Let $\gamma \in \mathrm{End}_{\mathcal{C}}(x)$ for a fixed object x. We show γ is an isomorphism. In fact, by the second condition in the definition of a category with subobjects, γ admits a factorization as $\gamma_0 f$, where $\gamma_0 \in \mathrm{Hom}_{\mathcal{I}}(x', x)$ and $f \in \mathrm{Is}_{\mathcal{C}}(x, x')$. Since $\mathcal{C}$ is skeletal, we have $x = x'$, hence $\gamma_0 \in \mathrm{Hom}_{\mathcal{I}}(x, x) = \mathrm{End}_{\mathcal{I}}(x) = \{1_x\}$, by the first condition of the definition of a category with subobjects. Then $\gamma = f$ is an isomorphism of x, and hence $\mathcal{C}$ is EI.

Also by the definition of a category with subobjects, when $\mathcal{C}$ is skeletal the following statement has to be true: if $x, y \in \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$, then $|\mathrm{Hom}_{\mathcal{I}}(x, y)| = 1$. This implies $\mathcal{I} \cong P(\mathcal{C})$. □

We give some useful characterizations of EI-categories with subobjects.

Lemma 4.6.5. *Let $(\mathcal{C}, \mathcal{I})$ be an EI-category with subobjects. Then*

1. *all morphisms in $\mathcal{C}$ are monomorphisms;*
2. *if $\mathcal{C}$ is a finite category with subobjects, then the number of objects $z \in [x]$ for which $\mathrm{Hom}_{\mathcal{I}}(z, y) \neq \emptyset$ equals $|\mathrm{Hom}_{\mathcal{C}}(x, y)|/|\mathrm{Aut}_{\mathcal{C}}(x)|$, for any $y \ncong x$ such that $\mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$;*
3. *if $\mathcal{C}$ is skeletal $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely and transitively on $\mathrm{Hom}_{\mathcal{C}}(x, y)$, for any pair of objects $x, y \in \mathrm{Ob}\,\mathcal{C}$.*

Proof. To prove 1, let $\alpha \in \mathrm{Hom}_{\mathcal{C}}(y, z)$ and $\gamma, \gamma' \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ satisfying $\alpha\gamma = \alpha\gamma'$. We show $\gamma = \gamma'$, and then α is monomorphic.

Since $\alpha = \alpha_0 f$ for some $\alpha_0 \in \mathrm{Hom}_{\mathcal{I}}(y', z)$ and $f \in \mathrm{Is}_{\mathcal{C}}(y, y')$ (where $y' \cong y$ in $\mathcal{C}$), we can write $\alpha\gamma = \alpha_0(f\gamma) = \alpha_0(\gamma_0 g)$ and $\alpha\gamma' = \alpha_0(f\gamma') = \alpha_0(\gamma'_0 g')$ where

$g \in \mathrm{Is}_{\mathcal{C}}(x, x'), \gamma_0 \in \mathrm{Hom}_{\mathcal{I}}(x', y'), g' \in \mathrm{Is}_{\mathcal{C}}(x, x''), \gamma_0' \in \mathrm{Hom}_{\mathcal{I}}(x'', y')$ for some $x' \cong x$ and $x'' \cong x$, which satisfy the equalities $f\gamma = \gamma_0 g$ and $f\gamma' = \gamma_0' g'$. By condition (2) in the definition of a category with subobjects and $(\alpha_0\gamma_0)g = (\alpha_0\gamma_0')g'$, we get $g = g'$ hence $\gamma_0 = \gamma_0'$ because we must have $x' = x''$ and as a consequence $\mathrm{Hom}_{\mathcal{I}}(x', y') = \mathrm{Hom}_{\mathcal{I}}(x'', y')$. So $\gamma = f^{-1}(\gamma_0 g) = f^{-1}(\gamma_0' g') = \gamma'$, which means α is a monomorphism.

Now suppose $\mathcal{C}$ is finite. Then statement 2 makes sense because every $[x]$ is finite and the morphism sets, such as $\mathrm{Hom}_{\mathcal{C}}(x, y)$, are always finite. Since $\mathcal{C}$ is EI by Proposition 4.6.3, $\mathrm{Aut}_{\mathcal{C}}(x)$ acts freely on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ due to part 1. There are $n = |\mathrm{Hom}_{\mathcal{C}}(x, y)|/|\mathrm{Aut}_{\mathcal{C}}(x)|$ orbits of morphisms. Let's write the orbits as $\overline{\alpha_1}, \cdots, \overline{\alpha_n}$, given by the representatives $\alpha_1, \cdots, \alpha_n \in \mathrm{Hom}_{\mathcal{C}}(x, y)$. Note that if $\alpha_i = \alpha_{i0} f$ for some $\alpha_{i0} \in \mathrm{Hom}_{\mathcal{I}}(x', y)$ and $f \in \mathrm{Is}_{\mathcal{C}}(x, x')$, then any other representative α_i', satisfying $\overline{\alpha_i} = \overline{\alpha_i'}$, can be written as $\alpha_i' = \alpha_{i0}(fg)$ where $g \in \mathrm{Aut}_{\mathcal{C}}(x)$ and $\alpha_i g = \alpha_i'$. We claim if $i \neq j$, then α_i and α_j must factor through distinct x' and x'' which are isomorphic to x. Let's say $\alpha_i = \alpha_{i0} f$ and $\alpha_j = \alpha_{j0} f'$. If $x' = x''$, then $\alpha_{i0} = \alpha_{j0}$. Hence $\alpha_i = \alpha_{i0} f = \alpha_{j0} f = \alpha_{j0} f'(f'^{-1} f) = \alpha_j(f'^{-1} f)$, a contradiction. Thus the set $A_{xy} = \{z \in [x] | \mathrm{Hom}_{\mathcal{I}}(z, y) \neq \emptyset\}$ has size at least $n = |\mathrm{Hom}_{\mathcal{C}}(x, y)|/|\mathrm{Aut}_{\mathcal{C}}(x)|$, and we show $|A_{xy}|$ cannot be larger than this number. Suppose $x', x'' \in A_{xy}$. Let $f \in \mathrm{Is}_{\mathcal{C}}(x, x')$, $f' \in \mathrm{Is}_{\mathcal{C}}(x, x'')$, $\alpha_{i0} \in \mathrm{Hom}_{\mathcal{I}}(x', y)$ and $\alpha_{j0} \in \mathrm{Hom}_{\mathcal{I}}(x'', y)$. Then the two morphism sets $\alpha_{i0} f \,\mathrm{Aut}_{\mathcal{C}}(x)$ and $\alpha_{j0} f' \,\mathrm{Aut}_{\mathcal{C}}(x)$ are mutually exclusive, because if $\alpha_{i0} f g = \alpha_{j0} f' g'$ for $g, g' \in \mathrm{Aut}_{\mathcal{C}}(x)$ we would obtain $\alpha_{i0} = \alpha_{j0} f' g' g^{-1} f^{-1}$, which is impossible. If we choose for every $x_i \in A_{xy}$ an isomorphism $f_i \in \mathrm{Is}_{\mathcal{C}}(x, x_i)$, and assume $\mathrm{Hom}_{\mathcal{I}}(x_i, y) = \{\alpha_{i0}\}$. Then $\mathrm{Hom}_{\mathcal{C}}(x, y)$ is a disjoint union of sets $\coprod_{A_{xy}} \alpha_{i0} f_i \,\mathrm{Aut}_{\mathcal{C}}(x)$, each of which has the same size $|\mathrm{Aut}_{\mathcal{C}}(x)|$. So we must have $|A_{xy}| = n$.

Part 3 is true by definition. □

Let $(\mathcal{C}, \mathcal{I})$ be a category with subobjects. Suppose $\mathcal{D} \subset \mathcal{C}$ is a full subcategory satisfying the condition that if $x \in \mathrm{Ob}\,\mathcal{D}$ then $[x] \subset \mathrm{Ob}\,\mathcal{D}$. We can naturally make $\mathcal{D}$ into a category with subobjects $(\mathcal{D}, \mathcal{I} \cap \mathcal{D})$. Then we can talk about full subcategories of a category with subobjects. In the rest of this section, we will consider the subposets $\mathrm{Wess}(\mathcal{I})$ and $\mathrm{Wess}_0(\mathcal{I})$ of $\mathcal{I}$. We show finite categories with subobjects have some similar homological properties to certain subposets of $\mathcal{I}$, and thus $\mathrm{Wess}_0(\mathcal{I})$ and $\mathrm{Wess}(\mathcal{I})$ may be used in a similar way as Bouc has done for posets. It's well-known that when $\mathcal{C}$ is a poset and $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is an inclusion, the overcategory $\iota \downarrow_y$ can be identified with the subposet $\mathcal{D}_{\leq y}$, and this is the key to prove a Bouc's result by Quillen's Theorem A. The next lemma generalizes this result to skeletal categories with subobjects. In Section 3.3 we've seen that for a pair of categories $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$, if $\mathcal{D}$ is a full subcategory and $y \in \operatorname{Ob}\mathcal{D}$, $\iota\downarrow_y$ is contractible because it has a terminal object $(y, 1_y)$. When $\mathcal{C}$ is an EI-category with subobjects and $y \in \operatorname{Ob}\mathcal{C}\setminus\operatorname{Ob}\mathcal{D}$, we can also say something about the overcategory $\iota\downarrow_y$.

Lemma 4.6.6. *Let $(\mathcal{C},\mathcal{I})$ be an EI-category with subobjects, and $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ a full subcategory. Then for any $y \in \operatorname{Ob}\mathcal{C}\setminus\operatorname{Ob}\mathcal{D}$, the skeleton of $\iota\downarrow_y$ is isomorphic to a poset, which can be identified with the poset $\mathcal{I}_{<y} \cap \mathcal{D}$. If $\mathcal{D}$ has a unique minimal object, then $\mathcal{I}_{<y} \cap \mathcal{D}$, if not empty, has an initial object and is contractible. If $\mathcal{C}$ is skeletal, we have $\mathcal{I}_{<y} \cap \mathcal{D} \cong P(\mathcal{D}_{<y})$.*

Proof. If we fix an $x \in \operatorname{Ob}\mathcal{D}_{<y}$ then every object $(x', \alpha') \in \iota\downarrow_y$ with $x' \cong x$ is isomorphic to some (x_i, α_i), where $x_i \cong x$ and $\alpha_i \in \operatorname{Hom}_{\mathcal{I}}(x_i, y)$. Since $(x_i, \alpha_i) \cong (x_j, \alpha_j)$ if and only if $x_i = x_j$, the skeleton of $\iota\downarrow_y$ is isomorphic to the full subcategory consisting of objects $\{(x, \alpha) | x \in \operatorname{Ob}\mathcal{D}_{<y}, \alpha \in \operatorname{Hom}_{\mathcal{I}}(x, y)\}$. Using the definition of a category with subobjects, it's easy to see the full subcategory is a poset, and is isomorphic to $\mathcal{I}_{<y} \cap \mathcal{D}$ by our assumption.

When $\mathcal{D}$ has a unique minimal object, so does $\mathcal{I}_{<y} \cap \mathcal{D}$. Hence it's contractible because in the poset the unique minimal object is indeed an initial object. If $\mathcal{C}$ is skeletal, every isomorphism class of objects contains only one object. So the identification $\mathcal{I}_{<y} \cap \mathcal{D} \cong P(\mathcal{D}_{<y})$ follows. □

Definition 4.6.7. *Let $(\mathcal{C},\mathcal{I})$ be a finite category with subobjects. Then we denote the two full subcategories of $\mathcal{C}$ which share the same object sets with $\mathrm{Wess}_0(\mathcal{I})$ and $\mathrm{Wess}(\mathcal{I})$, respectively, by $\mathcal{D}_{\mathrm{Wess}_0}$ and $\mathcal{D}_{\mathrm{Wess}}$. Obviously $\mathcal{D}_{\mathrm{Wess}_0} \subset \mathcal{D}_{\mathrm{Wess}}$.*

We comment here that in general $\mathrm{Wess}_0(\mathcal{C}) \subsetneq \mathcal{D}_{\mathrm{Wess}_0}$ and $\mathrm{Wess}(\mathcal{C}) \subsetneq \mathcal{D}_{\mathrm{Wess}}$. But when $(\mathcal{C},\mathcal{I})$ is skeletal, we do have $\mathrm{Wess}_0(\mathcal{C}) = \mathcal{D}_{\mathrm{Wess}_0}$ and $\mathrm{Wess}(\mathcal{C}) = \mathcal{D}_{\mathrm{Wess}}$. Our next two propositions show the importance of these two new full subcategories of $\mathcal{C}$.

Proposition 4.6.8. *Let $\mathcal{C}$ be a finite category with subobjects. Then $\mathcal{D}_{\mathrm{Wess}_0}$ is the smallest full subcategory among all full subcategories of $\mathcal{C}$, relative to which $\underline{R}$ is projective. Consequently, $\underline{R}$ is projective relative to $\mathcal{D}_{\mathrm{Wess}}$, and $\mathcal{V}_{\underline{R}}$ is the smallest convex subcategory (or ideal) of $\mathcal{C}$ that contains $\mathcal{D}_{\mathrm{Wess}_0}$.*

Proof. By Proposition 4.5.1 and Lemma 4.6.4, we know $\underline{R}$ is $\mathcal{D}_{\text{Wess}_0}$-projective. If $\mathcal{D}$ contains $\mathcal{D}_{\text{Wess}}$ then $\underline{R}$ is $\mathcal{D}$-projective. On the other hand if $\mathcal{D}_{\text{Wess}} \not\subset \mathcal{D}$, then by Lemma 4.6.4 again, there exists an overcategory associated with $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ which is disconnected. Thus $\underline{R}$ won't be $\mathcal{D}$-projective. □

Our next result is a generalization of Bouc's result on finite posets (see [5] Proposition 6.6.5, or [37] Proposition 3.10). Let $\mathcal{I}$ be a finite poset and $\mathcal{I}'$ a subposet. Bouc proved if $\text{Wess}(\mathcal{I}) \subset \mathcal{I}' \subset \mathcal{I}$ then the inclusions induce homotopy equivalences of the classifying spaces of these posets.

Proposition 4.6.9. *Let $(\mathcal{C}, \mathcal{I})$ be a finite category with subobjects. Given a full subcategory $\mathcal{D}$ with $\mathcal{D}_{\text{Wess}} \subset \mathcal{D}$, the inclusions $|\mathcal{D}_{\text{Wess}}| \subset |\mathcal{D}| \subset |\mathcal{C}|$ induce homotopy equivalences.*

Proof. Following Bouc's idea, we're going to use Quillen's Theorem A to prove $\mathcal{D}_{\text{Wess}} \subset \mathcal{D}$ induces an equivalence. Let's take any object $y \in \text{Ob}\,\mathcal{D}$. We want to show the overcategory category $\iota \downarrow_y$ is always contractible, where $\iota : \mathcal{D}_{\text{Wess}} \hookrightarrow \mathcal{D}$ is the inclusion. If $y \in \text{Ob}\,\mathcal{D}_{\text{Wess}}$, then this category has a terminal object $(y, 1_y)$, and hence is contractible. If $y \notin \text{Ob}\,\mathcal{D}_{\text{Wess}}$, we claim $\iota \downarrow_y$ is still contractible. By our assumption and Lemma 4.6.4, we know the skeleton of $\iota \downarrow_y$ is isomorphic to the poset $(\mathcal{I} \cap \mathcal{D})_{<y} \cap \text{Wess}(\mathcal{I})$, because $(\mathcal{D}_{\text{Wess}}, \text{Wess}(\mathcal{I})) \subset (\mathcal{D}, \mathcal{I} \cap \mathcal{D})$ is a full subcategory. Since $(\mathcal{I} \cap \mathcal{D})_{<y} \cap \text{Wess}(\mathcal{I}) = \text{Wess}(\mathcal{I}_{<y}) \subset \mathcal{I}_{<y}$ and $\mathcal{I}_{<y}$ is contractible by definition, from Bouc's original result we have $\text{Wess}(\mathcal{I})_{<y}$ contractible. Hence so is $\iota \downarrow_y$ and we're done. □

The above two propositions result in the following reduction of higher limits.

Corollary 4.6.10. *Let $(\mathcal{C}, \mathcal{I})$ be a category with subobjects and $\mathcal{D}$ a full subcategory satisfying either one of the following conditions:*

1. *$\mathcal{D}_{\text{Wess}} \subset \mathcal{D}$; or*

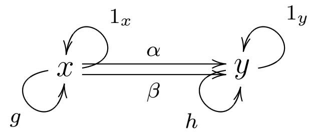
2. *$\mathcal{D}_{\text{Wess}_0} \subset \mathcal{D}$ and $R\mathcal{C}$ is a right flat $R\mathcal{D}$-module,*

then we have $\varprojlim^*_{\mathcal{C}} N \cong \varprojlim^*_{\mathcal{D}} N \downarrow^{\mathcal{C}}_{\mathcal{D}}$. Note that if $N = \underline{R}$, this isomorphism gives rise to a ring isomorphism.

Proof. If the condition in (1) is satisfied, then every overcategory associated to the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is contractible as is shown in Proposition 4.6.6. So we can use Corollary 3.2.6 to establish the isomorphisms.

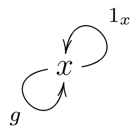
If the conditions in (2) are satisfied, then we can use Proposition 4.6.5 and Lemma 4.4.19 to get the isomorphism. $\square$

Example 4.6.11. *Let $R = \mathbb{F}_2$ and $\mathcal{C}$ be the following EI-category*



,

where $\alpha g = \beta$ and $\beta g = \alpha$. We're not going to use other morphisms since we can let y have an arbitrary automorphism group, which don't affect the higher limits. First of all, it's easy to see that $\mathcal{C}$ is a category with subobjects, where $\mathcal{I}$ can be chosen to be the poset $x \xrightarrow{\alpha} y$. Since $\mathrm{Wess}_0(\mathcal{I}) = \mathrm{Wess}(\mathcal{I})$ consists a single object x, $\mathcal{D}_{\mathrm{Wess}_0} = \mathcal{D}_{\mathrm{Wess}}$ are equal to following full subcategory.



Since $\mathcal{V}_{\underline{\mathbb{F}_2}}$ is the smallest convex subcategory containing $\{x\}$, $\mathcal{V}_{\underline{\mathbb{F}_2}} = \{x\}$, which can be verified to be $\mathbb{F}_2$-co-taut in $\mathcal{C}$. Then using the above corollary we deduce that $\varprojlim^n_{\{x\}} \downarrow^{\mathcal{C}}_{\{x\}} - \cong \varprojlim^n_{\mathcal{C}} -$, and as a special case we have $\varprojlim^n_{\{x\}} \underline{\mathbb{F}_2} \downarrow^{\mathcal{C}}_{\{x\}} \cong \varprojlim^n_{\mathcal{C}} \underline{\mathbb{F}_2} = \mathbb{F}_2$. It's interesting to note that $\mathrm{Wess}_0(\mathcal{C}) \subset \mathrm{Wess}(\mathcal{C}) = \mathcal{C}$.

Chapter 5

Resolutions and their applications

Using the results in last chapter we can establish an Eckmann-Shapiro type lemma

$$\mathrm{Ext}^*_{R\mathcal{C}}(M,N) \cong \mathrm{Ext}^*_{R\mathcal{C}_M}(M,N)$$

for any $R\mathcal{C}$-modules M and N, because $\mathcal{C}_M$ is a co-ideal in $\mathcal{C}$ relative to which M is projective. This can be shown in a more straightforward way: if $\mathcal{P} \to M \to 0$ is the minimal projective resolution, then every P_n should have $\mathcal{C}_{P_n} \subset \mathcal{C}_M$ which immediately results in the above isomorphism. In this chapter we study (minimal) projective resolutions of $R\mathcal{C}$-modules. We will see that not only the support of M but the support of N affects the computation of $\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$. As an example, we show how to use projective resolutions to give interesting interpretations of groups $\mathrm{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W})$. Finally, the characterization of the minimal projective resolutions of $R\mathcal{C}$-modules gives us some ideas about computing various homological dimensions.

5.1 The minimal projective resolution of an $R\mathcal{C}$-module

Fix an $M \in R\mathcal{C}$-mod, we describe its minimal projective resolution, which will help us to remove the objects beyond M-minimal and N-maximal objects which are "irrelevant with respect to computing $\mathrm{Ext}^*_{R\mathcal{C}}(M,N)$". One can check Definition 4.4.9 for the meanings of $\mathcal{C}_M$ and $\mathcal{C}^M$ for any $M \in R\mathcal{C}$-mod. As in last chapter, the base ring R is assumed to be a field or a complete discrete valuation ring.

Lemma 5.1.1. *Suppose M is an $R\mathcal{C}$-module. If $P_M \cong \oplus_{y,U} P_{y,U}$ is the projective cover of M, then every such y belongs to $\mathcal{C}_M$, which means that P_M is supported on $\mathcal{C}_M$. Thus $M/\mathrm{Rad}\, M$ is a semi-simple module which is isomorphic to a direct sum of some simple modules supported on $\mathcal{C}_M$.*

If $\mathcal{D}$ is an ideal in $\mathcal{C}$ such that $\mathcal{D} \cap \mathcal{C}_M \neq \emptyset$, then $P_M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ is the projective cover of $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$. Particularly, if x is an M-minimal object, then $P_M(x)$ is the projective cover of $M(x)$.

Proof. We only have to show that any indecomposable projective module $P_{y',U'}$, with $y' \notin \mathrm{Ob}\,\mathcal{C}_M$ and $\mathrm{Hom}_{\mathcal{C}}(y', x) \neq \emptyset$ for some M-minimal x, isn't isomorphic to a direct summand of P_M. If it were a summand, then the restriction of the defining (essential) surjection $\pi : P_M \to M$ satisfying the following commutative diagram for all $f \in \mathrm{Hom}(y', x)$

$$\begin{array}{ccc} P_{y',U'}(x) & \xrightarrow{\pi_x} & M(x) \\ {\scriptstyle P_{y',U'}(f)}\uparrow & & \uparrow{\scriptstyle M(f)} \\ P_{y',U'}(y') & \xrightarrow[\pi_{y'}]{} & M(y') = 0. \end{array}$$

It forces π_x to be the zero map. Hence π restricted on $P_{y',U'}$ is trivial which is a contradiction to the definition of the projective cover of a module.

Now let $\mathcal{D}$ be an ideal in $\mathcal{C}$ with $\mathcal{D} \cap \mathcal{C}_M \neq \emptyset$. Then by our result in last chapter $P_M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ is a projective $R\mathcal{D}$-module which admits a surjection onto $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$. So we just have to show $P_M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ is the smallest among modules satisfying these properties. Let $\pi : P_M \to M$ be the natural (essential) surjection. Then after restriction we obtain a surjection $\pi|_{\mathcal{D}} : P_M \downarrow^{\mathcal{C}}_{\mathcal{D}} \to M \downarrow^{\mathcal{C}}_{\mathcal{D}}$. Suppose $P_M \cong [\oplus_{x \in \mathrm{Ob}\,\mathcal{D}} P_{x,V}] \oplus [\oplus_{y \notin \mathrm{Ob}\,\mathcal{D}} P_{y,U}]$. Then by the structure theorem of projective modules $P_{y,U} \downarrow^{\mathcal{C}}_{\mathcal{D}} = 0$ and $P_{x,V} \downarrow^{\mathcal{C}}_{\mathcal{D}} = \widetilde{P_{x,V}}$, where $\widetilde{P_{x,V}}$ is a projective $R\mathcal{D}$-module whose values on $z \in \mathrm{Ob}\,\mathcal{D}$ are the same as those of $P_{x,V}$, and zero otherwise. If $P_M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ were not the projective cover of $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$, by the standard property of the projective cover of a module, $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\, P_M$ would be a direct summand $\widetilde{P_1} \oplus \widetilde{P_2}$, where P_1 and P_2 are two direct summands of P_M such that $\widetilde{P_1}$ is the projective cover of $\mathrm{Res}^{\mathcal{C}}_{\mathcal{D}}\, M$ and $\widetilde{P_2} \neq 0$. Since $\pi|_{\mathcal{D}}$ sends $\widetilde{P_1}$ surjectively onto $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ and $\widetilde{P_2}$ to 0, this implies $\pi(P_2) = 0$ because P_2 is generated by its value on some object in $\mathcal{D}$. Thus we obtain a contradiction to the minimality of P_M.

Now let x be an M-minimal object. Then $\mathcal{C}_{\leq x}$ is an ideal in $\mathcal{C}$ and $\mathcal{C}_{\leq x} \cap \mathcal{C}_M = \{[x]\} \neq \emptyset$. Hence $P_M \downarrow^{\mathcal{C}}_{\mathcal{C}_{\leq x}}$ is a projective cover of $M \downarrow^{\mathcal{C}}_{\mathcal{C}_{\leq x}}$. But x is a maximal object in $\mathcal{C}_{\leq x}$. We get immediately $P_M(x)$ is the projective cover of $M(x)$. $\square$

With the above lemma we can go on to describe the minimal projective resolution of an arbitrary $R\mathcal{C}$-module. Given an $R\mathcal{C}$-module M and its projective cover P_M, from the previous lemma we know $P_M \cong \oplus_{y,U} P_{y,U}$ with $y \in \mathrm{Ob}\,\mathcal{C}_M$. When we look at the minimal resolution of M

$$\mathcal{P}_M : \cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0,$$

we know P_0 is simply P_M and P_1 is the projective cover of K_0, the kernel of the map $P_0 = P_M \to M$. Since $\mathcal{C}_{K_0} \subset \mathcal{C}_{P_0} \subset \mathcal{C}_M$, we have $\mathcal{C}_{P_1} \subset \mathcal{C}_M$ too by the lemma. Hence we conclude the following result on the minimal projective resolution by repeating the same argument for every P_n.

Corollary 5.1.2. *Let M be an $R\mathcal{C}$-module and $\mathcal{P}_M$ its minimal projective resolution. Then $\mathcal{P}_M$ is supported on $\mathcal{C}_M$. Suppose $\mathcal{D}$ is an ideal in $\mathcal{C}$. Then $\mathcal{P}_M \downarrow^{\mathcal{C}}_{\mathcal{D}}$ is the minimal projective resolution of $M \downarrow^{\mathcal{C}}_{\mathcal{D}}$.*

If M is an atomic module whose support is $\{[x]\}$ for some $x \in \mathrm{Ob}\,\mathcal{C}$. Then in the minimal projective resolution of M, $P_1 \cong \oplus_{y,U} P_{y,U}$ with $y \cong x$ or $\mathrm{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$ if $y \ncong x$.

Proof. The first statement is a direct consequence of the preceding lemma and the discussion after it, and so we prove the second part here. Indeed, it's earlier to see this using a larger projective resolution of M. Suppose $\widetilde{\mathcal{P}_M}$ is constructed in this way: $\widetilde{P_0} = (R\mathcal{C} \cdot 1_x)^n$ for some positive integer n, and $\widetilde{P_1}$ is the projective cover of the kernel $\widetilde{K_0}$ of the map $\widetilde{P_0} \to M \to 0$. Then $\widetilde{K_0}$ contains all non-isomorphisms in $\widetilde{P_0}$, and thus $\mathrm{Rad}(R\mathcal{C}) \cdot \widetilde{K_0} = \mathrm{Rad}(\widetilde{K_0})$ contains all reducible morphisms in $\widetilde{P_0}$. This implies $\widetilde{P_1}$, as the projective cover of $\widetilde{K_0}$, is isomorphic to a direct sum of the form $\oplus_{y,U} P_{y,U}$, where $y \cong x$ or $y \ncong x$ and $\mathrm{Irr}_{\mathcal{C}}(x, y) \neq \emptyset$. Since $\mathcal{P}_M$ is the minimal projective resolution of M, $\mathcal{P}_M$ must be isomorphic to a direct summand of $\widetilde{\mathcal{P}_M}$. In particular, P_1 is isomorphic to a direct summand of $\widetilde{P_1}$. Hence the statement follows. $\square$

Using our results on relative projectivity, Lemma 5.1.1 can be strengthened.

Proposition 5.1.3. *Let M be an $R\mathcal{C}$-module which is $\mathcal{D}$-projective for a full subcategory $\mathcal{D}$. Then $M/\mathrm{Rad}(M)$ has a support contained in $\mathcal{D}$. As a consequence, the projective cover P_M is $\mathcal{D}$-projective.*

proof. Since M is $\mathcal{D}$-projective, for any $y \in \operatorname{Ob}\mathcal{C} \setminus \operatorname{Ob}\mathcal{D}$, we have

$$M(y) = \sum_{x \in \operatorname{Ob}\mathcal{D}_{\leq y}} R\operatorname{Hom}_{\mathcal{C}}(x, y) \cdot M(x) \subset R\mathcal{C} \cdot M = \operatorname{Rad}(M).$$

Hence $M/\operatorname{Rad}(M)$ has a support contained in $\mathcal{D}$. This implies $P_M \cong \oplus_{y \in \operatorname{Ob}\mathcal{D}} P_{y,U}$, and then P_M is $\mathcal{D}$-projective because $\mathcal{D}$ contains the vertex of every $P_{y,U}$. □

Note that if M is indecomposable, then M and P_M usually have different vertices. One can consider $P_{x,V} \to S_{x,V}$ when they aren't equal. From the same example we can see the above proposition cannot be strengthened: M being $\mathcal{D}$-projective doesn't imply $M/\operatorname{Rad} M$ is $\mathcal{D}$-projective. Furthermore the kernel of the surjection $P_M \to M$ does not have to be $\mathcal{D}$-projective if M is. We can construct a category $\mathcal{C}$ with two objects x and y, in which $\operatorname{Aut}_{\mathcal{C}}(x) \cong C_2$, $\operatorname{Aut}_{\mathcal{C}}(y) = \{1_y\}$ and $\operatorname{Hom}_{\mathcal{C}}(x, y) = \{\alpha\}$. Suppose $R = \mathbb{F}_2$. Then the trivial module $\underline{\mathbb{F}_2}$ is $\{x\}$-projective and has projective cover $P_{x,1} = \mathbb{F}_2\mathcal{C} \cdot 1_x$. The kernel of the map $\mathbb{F}_2\mathcal{C} \cdot 1_x \to \underline{\mathbb{F}_2}$ is isomorphic to $(\mathbb{F}_2)_x$, a dimension 1 atomic module concentrated on x, whose vertex is equal to $\mathcal{C}$, not $\{x\}$. This example implies that, given a $\mathcal{D}$-projective module M and its minimal resolution $\mathcal{P}_M \to M \to 0$, we cannot expect all P_n to be $\mathcal{D}$-projective, except that $\mathcal{D} = \mathcal{C}_M$.

Next we show that $\mathcal{C}_M$ and $\mathcal{C}^N$ have some impact on computing $\operatorname{Ext}^*_{R\mathcal{C}}(M, N)$.

Proposition 5.1.4. *Given two $R\mathcal{C}$-modules M and N, we have*

$$\operatorname{Ext}^*_{R\mathcal{C}}(M, N) = \operatorname{Ext}^*_{R\mathcal{C}^N_M}(M \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}, N \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}),$$

where $\mathcal{C}^N_M = \mathcal{C}_M \cap \mathcal{C}^N$.

Proof. We take the minimal $R\mathcal{C}$-resolution of M

$$\mathcal{P}_M : \cdots \to P_1 \to P_0 \to M \to 0.$$

It is supported on $\mathcal{C}_M$ hence an $R\mathcal{C}_M$-resolution of $M \downarrow^{\mathcal{C}}_{\mathcal{C}_M}$. It's obvious that

$$\operatorname{Hom}_{R\mathcal{C}}(P_n, N) = \operatorname{Hom}_{R\mathcal{C}^N_M}(P_n \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}, N \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}),$$

and it passes to an isomorphism of cochain complexes

$$\{\operatorname{Hom}_{R\mathcal{C}}(\mathcal{P}_M, N)\} \cong \{\operatorname{Hom}_{R\mathcal{C}^N_M}(\mathcal{P}_M \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}, N \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M})\}.$$

If we can show $\mathcal{P}_M \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}$ is still a projective resolution of $M \downarrow^{\mathcal{C}}_{\mathcal{C}^N_M}$ as an $R\mathcal{C}^N_M$-module then we're done. But this comes from Lemma 5.1.1 since $\mathcal{C}^N_M$ is an ideal in $\mathcal{C}_M$. □

Corollary 5.1.5. *Let* $\mathcal{C}_x^y = \mathcal{C}_{\geq x} \cap \mathcal{C}_{\leq y}$. *Then*

$$\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W}) \cong \operatorname{Ext}^*_{R\mathcal{C}_x^y}(S_{x,V}, S_{y,W}).$$

In particular we have $\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{x,W}) \cong \operatorname{Ext}^*_{R\operatorname{Aut}(x)}(V, W)$.

Note that in the above corollary if $\operatorname{Hom}(x, y) = \emptyset$ then $\mathcal{C}_x^y = \emptyset$ hence $\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W})$ vanish.

Proposition 5.1.6. *Let* $S_{x,V}$ *and* $S_{y,W}$ *be two simple* $R\mathcal{C}$*-modules. If* $\operatorname{Irr}(x, y) = \emptyset$, *then* $\operatorname{Ext}^1_{R\mathcal{C}}(S_{x,V}, S_{y,W}) = 0$.

Proof. Suppose $\cdots \to P_1 \to P_0 \to S_{x,V} \to 0$ is the minimal projective resolution of $S_{x,V}$. Then since $S_{y,W}$ is simple, we have $\operatorname{Ext}^1_{R\mathcal{C}}(S_{x,V}, S_{y,W}) \cong \operatorname{Hom}(P_1, S_{y,w})$. But $P_1 \cong \oplus P_{z,V}$ for $z \cong x$ or $z \in \operatorname{Ob}\mathcal{C}_{\geq x}$ such that $\operatorname{Irr}(x, z) \neq \emptyset$. By our assumption the degree one Ext group has to be zero because $P_{y,W}$ cannot be a direct summand of P_1. □

5.2 Computing $\operatorname{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R})$

As an interesting example, we try to calculate the groups $\operatorname{Ext}^*_{R\mathcal{C}}(S_{x,V}, S_{y,W})$. By Corollary 5.1.4, we only need to deal with the situation when $\operatorname{Hom}_{\mathcal{C}}(x, y) \neq \emptyset$. When the two simples are $S_{x,R}$ and $S_{x,R}$, we can give rather explicit interpretations of the Ext groups. For simplicity, in this section we assume $\mathcal{C}$ is skeletal and has x as the unique minimal object and y as the unique maximal object, i.e. $\mathcal{C} = \mathcal{C}_x^y$.

Consider the trivial $R\mathcal{C}$-module $\underline{R}$. It has a unique maximal submodule, denoted by $\underline{R}_{>x}$, which takes value 0 at x, and R elsewhere. The quotient module is obviously isomorphic to the simple module $S_{x,R}$. So we have a short exact sequence of $R\mathcal{C}$-modules

$$0 \to \underline{R}_{>x} \xrightarrow{i} \underline{R} \to S_{x,R} \to 0.$$

It gives rise to a long exact sequence (a)

$$\begin{array}{lllllll} 0 & \to & \operatorname{Hom}(S_{x,R}, S_{y,R}) & \to & \operatorname{Hom}(\underline{R}, S_{y,R}) & \xrightarrow{i^*} & \operatorname{Hom}(\underline{R}_{>x}, S_{y,R}) \\ & \to & \operatorname{Ext}^1(S_{x,R}, S_{y,R}) & \to & \operatorname{Ext}^1(\underline{R}, S_{y,R}) & \xrightarrow{i^*} & \operatorname{Ext}^1(\underline{R}_{>x}, S_{y,R}) \\ & & \cdots & & \cdots & & \cdots \\ & \to & \operatorname{Ext}^n(S_{x,R}, S_{y,R}) & \to & \operatorname{Ext}^n(\underline{R}, S_{y,R}) & \xrightarrow{i^*} & \operatorname{Ext}^n(\underline{R}_{>x}, S_{y,R}) \\ & & \cdots & & \cdots & & \cdots. \end{array}$$

We're going to use this long exact sequence to calculate the two groups $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, M)$ and $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}_{>x}, M)$ over some special category $\mathcal{C}$.

Proposition 5.2.1. *Let $\mathcal{C} = \mathcal{C}_x^y$ be a connected EI-category, where $x \not\cong y$. Suppose $\underline{R}$ is a projective $R\mathcal{C}$-module. Then we have $\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, M) \cong \varprojlim^{n-1}_{\mathcal{C}_{>x}} M \downarrow^{\mathcal{C}}_{\mathcal{C}_{>x}}$ for $n \geq 2$ and $\iota : \mathcal{C}_{>x} \hookrightarrow \mathcal{C}$.*

If $\underline{R}$ is a projective $R\mathcal{C}^{op}$-module then we have $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, S_{y,R}) \cong \tilde{\mathrm{H}}^{n-1}(|\mathcal{C}_{<y}|, R)$ for $n \geq 1$, and $\mathrm{Ext}^0_{R\mathcal{C}}(\underline{R}, S_{y,R}) = 0$.

Proof. From the preceding long exact sequence we get

$$\mathrm{Ext}^{n-1}_{R\mathcal{C}}(\underline{R}_{>x}, M) \cong \mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, M)$$

when $n \geq 2$. Then by Proposition 3.1.4 we have $\mathrm{Ext}^{n-1}_{R\mathcal{C}}(\underline{R}_{>x}, M) \cong \varprojlim^{n-1}_{\mathcal{C}_{>x}} \mathrm{Res}_\iota M$.

Next we consider the long exact sequence derived from the sequence of $R\mathcal{C}$-modules $0 \to S_{y,R} \to \underline{R} \to \underline{R}_{\mathcal{C}_{<y}} \to 0$ by applying $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, -)$, where $\underline{R}_{<y}$ is the quotient $\underline{R}/S_{y,R}$. In the long exact sequence we'll have $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, \underline{R}) = 0$ for $n \geq 1$ because $\underline{R}$ is a projective $R\mathcal{C}^{op}$-module and $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong \mathrm{Ext}^n_{R\mathcal{C}^{op}}(\underline{R}, \underline{R})$. Thus $\mathrm{Ext}^n_{R\mathcal{C}}(\underline{R}, S_{y,R}) \cong \mathrm{Ext}^{n-1}_{R\mathcal{C}}(\underline{R}, \underline{R}_{<y}) \cong \mathrm{H}^{n-1}(|\mathcal{C}_{<y}|, R)$ for $n \geq 2$. For $n = 0$, we look at the exact sequence

$$0 \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, S_{y,R}) \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, \underline{R}) \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, \underline{R}_{<y}) \to \mathrm{Ext}^1_{R\mathcal{C}}(\underline{R}, S_{y,R}) \to 0,$$

and it's easy to see $\mathrm{Ext}^0_{R\mathcal{C}}(\underline{R}, S_{y,R}) \cong \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, S_{y,R}) = 0$ and $\mathrm{Ext}^1_{R\mathcal{C}}(\underline{R}, S_{y,R}) = 0$. ∎

Suppose $\mathcal{C} = \mathcal{C}_x^y$. We use $\mathcal{C}_{(x,y)}$ to denote the full subcategory of $\mathcal{C}$ after removing $[x]$ and $[y]$.

Corollary 5.2.2. *Suppose $\mathcal{C} = \mathcal{C}_x^y$ and $\underline{R}$ is projective as both an $R\mathcal{C}$-module and an $R\mathcal{C}^{op}$-module. Then*

1. *if $\mathcal{C}_{(x,y)} \neq \emptyset$ then for $n \geq 2$, $\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R}) \cong \tilde{\mathrm{H}}^{n-2}(|\mathcal{C}_{(x,y)}|, R)$ and*
$$\mathrm{Ext}^0_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = \mathrm{Ext}^1_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0;$$
2. *if $\mathcal{C}_{(x,y)} = \emptyset$ we have*
$$\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R}) \cong \mathrm{Ext}^{n-1}_{R\,\mathrm{Aut}_{\mathcal{C}}(x)}(R, R) \cong \mathrm{Ext}^{n-1}_{R\,\mathrm{Aut}_{\mathcal{C}}(y)}(R, R)$$
when $n \geq 1$, while $\mathrm{Ext}^0_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0$.

Proof. By the preceding proposition, for $n \geq 2$ we have

$$\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R}) \cong \varprojlim{}^{n-1}_{\mathcal{C}_{>x}} S_{y,R},$$

and the latter turns out to be isomorphic to $\mathrm{Ext}^{n-1}_{\mathcal{C}_{(x,y)}}(\underline{R}, \underline{R})$, where $\mathcal{C}_{(x,y)} = (\mathcal{C}_{>x})_{<y}$. Now for $n = 0$ we have $\mathrm{Ext}^0_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = \mathrm{Hom}_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0$. For $n = 1$ from the exact sequence

$$0 \to \mathrm{Hom}_{R\mathcal{C}}(S_{x,R}, S_{y,R}) \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}, S_{y,R}) \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}_{>x}, S_{y,R}) \to \mathrm{Ext}^1_{R\mathcal{C}}(S_{x,R}, S_{y,R}) \to 0,$$

we know $\mathrm{Ext}^1_{R\mathcal{C}}(S_{x,R}, S_{y,R})$ is the cokernel of the natural map $\mathrm{Hom}_{R\mathcal{C}}(\underline{R}, S_{y,R}) \to \mathrm{Hom}_{R\mathcal{C}}(\underline{R}_{>x}, S_{y,R})$, which is 0.

The second statement can be proved in a similar way. □

We comment that Corollary 5.2.2 does not apply to categories with a unique minimal object and a unique maximal object where the automorphism groups do not act transitively.

Example 5.2.3. *We consider the category* $\mathcal{C}$

$$x \xrightarrow{\alpha} z \underset{\gamma}{\overset{\beta}{\rightrightarrows}} w \xrightarrow{\mu} y\ .$$

If $\beta\alpha = \gamma\alpha$ *then* $\mathcal{C}$ *has* x *as the initial object, and if* $\mu\beta = \mu\gamma$ *then* y *is the terminal object. When these two conditions are satisfied,* $\underline{R}$ *is projective as both an* $R\mathcal{C}$*- and an* $R\mathcal{C}^{op}$*-module. Therefore we can apply Corollary 5.2.2 to compute* $\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R})$.

However, if either x *isn't initial or* y *isn't terminal, the statements of Corollary 5.2.2 are no longer correct. In fact we can easily find it's always true that* $\mathrm{Ext}^3_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0$ *and* $\tilde{\mathrm{H}}^1(|\mathcal{C}_{(x,y)}|, R) = R$ *for any field* R.

When $\mathcal{C}$ is a poset, every subposet $\mathcal{C}^y_x$ satisfies the conditions in Corollary 5.2.2, and this gives rise to our next corollary, which is the main theorem of Igusa and Zacharia [22] where they used the theorem of Gerstenhaber and Shack [16] (quoted as Theorem 3.2.5 here in this thesis) which identifies the Hochschild cohomology and simplicial cohomology of posets.

Corollary 5.2.4. *Let* $\mathcal{C}$ *be a poset and* $x < y \in \mathrm{Ob}\,\mathcal{C}$. *Then* $\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = \tilde{\mathrm{H}}^{n-2}(|\mathcal{C}_{(x,y)}|, R)$ *for* $n \geq 2$ *if* $\mathcal{C}_{(x,y)} \neq \emptyset$, *and* $\mathrm{Ext}^0_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = \mathrm{Ext}^1_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0$. *If* $\mathcal{C}_{(x,y)} = \emptyset$ *then* $\mathrm{Ext}^1_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = R$ *and* $\mathrm{Ext}^n_{R\mathcal{C}}(S_{x,R}, S_{y,R}) = 0$ *for any* $n \neq 1$.

Note that in Corollary 5.2.4 when $\mathcal{C}_{(x,y)}$ is not empty then $\mathrm{Irr}_{\mathcal{C}}(x,y)=\emptyset$ because x is the initial and y is the terminal object in $\mathcal{C}_x^y$. One can compare this with Proposition 5.1.6.

5.3 Homological dimensions

Our work in the previous sections can be used to study the homological dimension of $R\mathcal{C}$-modules. With the description of the minimal resolution of any $R\mathcal{C}$-module we're able to give a theorem on the global dimension of $R\mathcal{C}$.

Recall that there is a poset $P(\mathcal{C})$ associated to every EI-category $\mathcal{C}$. When $\mathcal{C}$ is finite we call the maximal length of chains of non-isomorphisms in $P(\mathcal{C})$ the length of $\mathcal{C}$, denoted by $l(\mathcal{C})$.

Theorem 5.3.1. *Let $\mathcal{C}$ be a finite EI-category. Then $R\mathcal{C}$ has finite global dimension if and only if for all $x \in \mathrm{Ob}\,\mathcal{C}$, $|\mathrm{Aut}(x)|^{-1} \in R$. In fact,* $\mathrm{proj.dim}(M) \leq l(\mathcal{C}_M)$*, the length of $\mathcal{C}_M$. Hence* $\mathrm{gl.dim}(R\mathcal{C}) \leq l(\mathcal{C})$.

Proof. Since $R\mathcal{C}$ has finitely many simple modules, $R\mathcal{C}$ having finite global dimension is equivalent to the statement that every simple $R\mathcal{C}$-module has a finite projective resolution.

Suppose $|\mathrm{Aut}(x)|^{-1} \in R$ for all $x \in \mathrm{Ob}\,\mathcal{C}$. We do an induction on $l(\mathcal{C})$, the length of $\mathcal{C}$. Fix a simple $S_{y,W}$. It has a minimal projective resolution $\mathcal{P}$ written as

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to S_{y,W} \to 0.$$

Since $\mathcal{P}(y) \to S_{y,W}(y) = W \to 0$ is the minimal projective resolution of W and $|\mathrm{Aut}(y)|^{-1} \in R$, we have $P_0(y) \cong W$ for $R\,\mathrm{Aut}(y)$ is semi-simple. Then it implies that all $P_n, n>0$ are supported on $\mathcal{C}_{>y}$ for $P_n(y)=0$ when $n>0$. Therefore if we take K_0 as the kernel of the map $P_0 \to S_{y,W}$, K_0 must be an $R\mathcal{C}$-module supported on $\mathcal{C}_{>y}$, and

$$\cdots \to P_n \to \cdots P_1 \to K_0 \to 0$$

becomes a minimal projective resolution of K_0. Because $\mathcal{C}_{>y}$ has smaller length than $\mathcal{C}$, the resolution of K_0 is finite. So is $\mathcal{P}$, the resolution of $S_{y,W}$.

On the other hand if any $S_{y,W}$ has a finite projective resolution $\mathcal{P}$, then $\mathcal{P}(y)$ is a finite resolution of the simple $R\,\mathrm{Aut}(y)$-module $S_{y,W}(y)=W$. Since projective

$R\,\mathrm{Aut}(y)$-modules are the same as injective $R\,\mathrm{Aut}(x)$-modules, the finite exact sequence $\mathcal{P}(y) \to W \to 0$ splits. Hence each and every W is projective which means $R\,\mathrm{Aut}(y)$ is semi-simple, or equivalently $|\,\mathrm{Aut}(y)|^{-1} \in R$.

Under the circumstance if we consider an arbitrary $R\mathcal{C}$-module M, we can show proj.dim$(M) \le l(\mathcal{C}_M)$. Let

$$\mathcal{P}_M : 0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be the minimal projective resolution of M. Then P_0 is supported on $\mathcal{C}_M$. Just like what we've done above, it's easy to see P_1 is supported on $\mathcal{C}_M \backslash \{$M-minimal objects$\} \subset \mathcal{C}_M$, thus the size of $\mathcal{C}_{P_i}$ decreases strictly when i grows bigger and bigger. This means proj.dim$(M) \le l(\mathcal{C}_M)$. Hence gl.dim$(R\mathcal{C}) \le l(\mathcal{C})$. $\square$

Abusing the notation, when R is understood we will say $\mathcal{C}$ has finite global dimension if $R\mathcal{C}$ has finite global dimension.

Corollary 5.3.2. *Let $\mathcal{D}, \mathcal{C}$ be two EI-categories equipped with a functor $\iota : \mathcal{D} \to \mathcal{C}$. Suppose for any $x \in \mathrm{Ob}\,\mathcal{D}, y \in \mathrm{Ob}\,\mathcal{C}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\iota(x), y)$, $|\iota^{-1}(\mathrm{Stab}_{\mathrm{Aut}(\iota(x))}(\alpha))|$ is invertible in R. Then every overcategory $\iota \downarrow_y$ has finite global dimension.*

Proof. Fixing a $y \in \mathrm{Ob}\,\mathcal{D}$. Then objects of $\iota \downarrow_y$ are of the form (x, α) for some $x \in \mathrm{Ob}\,\mathcal{D}$ and $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\iota(x), y)$. The automorphism group of such an object is $\{g \in \mathrm{Aut}_{\mathcal{D}}(x) \mid \alpha\iota(g) = \alpha\}$, which equals $\iota^{-1}(\mathrm{Stab}_{\mathrm{Aut}(\iota(x))}(\alpha))$. Hence the result follows from the previous theorem. $\square$

When $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$ is an inclusion, the stabilizer group becomes $\mathrm{Stab}_{\mathrm{Aut}_{\mathcal{D}}(x)}(\alpha)$. If each group has an order invertible in R, then $\iota \downarrow_y$ has finite global dimension. Recall in Section 3.3 we studied the exactness of $R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P}_{\mathcal{D}}$ through the contractibility of overcategories. The above result says that if every stabilizer group has an order invertible in R, then the homology of $R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P}_{\mathcal{D}}$ vanish after certain stage, though $R\mathcal{C} \otimes_{R\mathcal{D}} \mathcal{P}_{\mathcal{D}}$ isn't necessarily exact.

In the same spirit we can give the following statement. When $\mathcal{D} \subset \mathcal{C}$ is a subcategory and $\mathrm{Mor}(\mathcal{C}) \backslash \mathrm{Mor}(\mathcal{D})$ contains only monomorphisms, all the stabilizers in the previous corollary are trivial, and hence the overcategories are of finite global dimension. As a special case we can take a finite category with subobjects $(\mathcal{C}, \mathcal{I})$. Then every morphism is monomorphic and hence the overcategories associated to the inclusion $\iota : \mathcal{D} \hookrightarrow \mathcal{C}$, of any subcategory $\mathcal{D}$, are equivalent to disjoint unions of finite posets, which have lengths less than or equal to that of $\mathcal{C}$.

Chapter 6

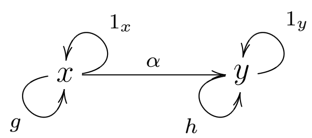
Cohomology rings of finite EI-categories

We already know the cohomology ring of a finite EI-category is the same as the cohomology ring of the topological realization of the category. Inspired by the known results for finite posets and finite groups, whose cohomology rings are finitely generated (see Evens [12] and Venkov [39]), in this chapter we look for further structure of the cohomology ring of an EI-Category. We note here that the finite generation of the cohomology ring of $\mathcal{C}$ does not immediately appear to follow from the result of Friedlander and Suslin [13] that the cohomology ring of a cocommutative Hopf algebra is finitely generated, because a category algebra usually isn't a cocommutative Hopf algebra.

6.1 The cohomology ring is not finitely-generated

The cohomology ring of an arbitrary finite EI-category is not finitely generated. Below is an example.

We consider the following category $\mathcal{C}$ over $\mathbb{F}_2$:



,

where $\alpha \cdot g = \alpha = h \cdot \alpha$, $g^2 = 1_x$ and $h^2 = 1_y$.

We're going to use the simplicial complex associated to $\mathcal{C}$ for computations:

$$\cdots \to C_n \to \cdots \to C_2 \to C_1 \to C_0 \to 0,$$

where C_0 is the $\mathbb{F}_2$-vectorspace generated by the objects, i.e., x and y, and C_n is the $\mathbb{F}_2$-vectorspace having a basis consisting of all chains made by n consecutive morphisms. For instance, C_2 has base elements $x\xrightarrow{g}x\xrightarrow{g}x$, $x\xrightarrow{g}x\xrightarrow{\alpha}y$ and $x\xrightarrow{\alpha}y\xrightarrow{h}h$. For brevity, we will simply denote them by $(g,g),(g,\alpha),(\alpha,h)$ and (h,h). Note that we can and do skip all the degenerated chains like $(1_x,g)$ and $(\alpha,1_y)$, because they don't contribute to our calculations.

The differential is given as follows

$$d(f_1,f_2,\cdots,f_n) = (f_2,\cdots,f_n) + \sum_{i\geq 1}^{n-1}(-1)^i(f_1,\cdots,m_i\cdot f_{i+1},\cdots,f_n)$$
$$+(-1)^n(f_1,f_2,\cdots,f_{n-1}),$$

where $(f_1,f_2,\cdots,f_n)$ is an element of C_n.

Now we pass to the dual complex in order to compute cohomology:

$$\cdots \leftarrow (C_n)^* \leftarrow \cdots \leftarrow (C_2)^* \leftarrow (C_1)^* \leftarrow (C_0)^* \leftarrow 0,$$

where $(C_n)^*$ is the $\mathbb{F}_2$-vectorspace generated by the dual base of C_n, i.e. $\{\delta_{(m_1,m_2,\cdots,m_n)}\}$, the set of functions indexed by $\{(m_1,m_2,\cdots,m_n)\}\subset C_n$ such that

$$\delta_{(m_1,m_2,\cdots,m_n)}(m_1',m_2',\cdots,m_n') = 0$$

unless $(m_1',m_2',\cdots,m_n') = (m_1,m_2,\cdots,m_n)$, whence the value of the function equals 1.

For $n>0$, C_n^* has base elements of the following types $\delta_{(g,\cdots,g)}$, $\delta_{(h,\cdots,h)}$, $\delta_{(g,\cdots,g,\alpha)}$, $\delta_{(\alpha,h,\cdots,h)}$ and $\delta_{(g,\cdots,g,\alpha,h,\cdots,h)}$. Direct calculations show that the cocycles $\delta_{(g,\cdots,g,\alpha)}$, $\delta_{(\alpha,h,\cdots,h)}\in C_{n+1}^*$ are the images of $\delta_{(g,\cdots,g)}$ and $\delta_{(h,\cdots,h)}\in C_n^*$ respectively. Since the differential restricted to all $\delta_{(g,\cdots,g,\alpha,h,\cdots,h)}$ is the zero map, the cohomology groups for $n>0$ is actually generated by all $\delta_{(g,\cdots,g,\alpha,h,\cdots,h)}$ with at least one g and one h on each end. Therefore we get $\mathrm{Ext}^n_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2}) = \mathbb{F}_2^{n-2}$ for $n\geq 3$, and $\mathrm{Ext}^1_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2}) = \mathrm{Ext}^2_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2}) = 0$.

Since the cup product of arbitrary two elements in $\oplus_{i\geq 1}\mathrm{Ext}^i_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2})$ is 0, the cohomology ring $\mathrm{Ext}^*_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2})$ is not finitely generated.

6.2 The cohomology ring modulo a nilpotent ideal

However, the above example is not the end of our story, because all elements in $\oplus_{i\geq 1}\operatorname{Ext}^i_{\mathbb{F}_2\mathcal{C}}(\underline{\mathbb{F}_2},\underline{\mathbb{F}_2})$ are can be easily seen to be nilpotent, and the cohomology ring modulo these nilpotents is indeed finitely generated! Nevertheless, this property stays true in all other examples we've computed so far. Hence the observation leads to the following conjecture.

Conjecture For a finite EI-category, its cohomology ring modulo nilpotents is finitely generated.

At the first sight one might think of the short exact sequence of complexes

$$0 \to \mathbb{A} \to \mathbb{C} \to \mathbb{C}/\mathbb{A} \to 0,$$

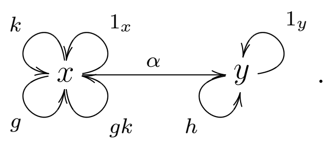
where $\mathbb{A}$ is the subcomplex of $\mathbb{C}$ generated by all automorphisms in $\mathcal{C}$. The short exact sequence gives rise to a long exact sequence which contains all necessary ingredients

$$\cdots \to \mathrm{H}^{n-1}(\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}/\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}) \to \mathrm{H}^n(\mathbb{A}) \to \mathrm{H}^{n+1}(\mathbb{C}/\mathbb{A}) \to \cdots.$$

If the maps $\mathrm{H}^{n-1}(\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}/\mathbb{A})$ were all zero, then the long exact sequence would break up to some short exact sequences like the following

$$0 \to \mathrm{H}^n(\mathbb{C}/\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}) \to \mathrm{H}^n(\mathbb{A}) \to 0.$$

Then if the image of $\mathrm{H}^n(\mathbb{C}/\mathbb{A})$ in $\mathrm{H}^n(\mathbb{C})$ were always nilpotent, we could prove our conjecture is correct. However, the break-down is not possible in general. For instance, the long exact sequence of the following category doesn't break into the desirable pieces.



A segment of the associated long exact sequence is recorded below.

$$\begin{array}{llllll} 0 & \to & 0 & \to & \mathbb{F}_2 & \to & \mathbb{F}_2^2 \\ & \to & \mathbb{F}_2 & \to & 0 & \to & \mathbb{F}_2^3 \\ & \to & \mathbb{F}_2^3 & \to & 0 & \to & \mathbb{F}_2^4 \\ & \to & \mathbb{F}_2^6 & \to & \mathbb{F}_2^2 & \to & \mathbb{F}_2^5 \\ & \to & \mathbb{F}_2^{10} & \to & \mathbb{F}_2^5 & \to & \mathbb{F}_2^6. \end{array}$$

For $n \geq 1$, we actually have some short exact sequences like

$$0 \to \mathrm{H}^n(\mathbb{A}) \to \mathrm{H}^{n+1}(\mathbb{C}/\mathbb{A}) \to \mathrm{H}^{n+1}(\mathbb{C}) \to 0,$$

which do not agree with the proposed break-down of the long exact sequence, and are not very helpful in describing the structure of the cohomology ring. However, the use of $\mathbb{A}$ helps us make a certain connection between the cohomology of $\mathcal{C}$ and cohomology of the automorphism groups insider it.

Proposition 6.2.1. *The image of* $\mathrm{H}^*(\mathbb{C}/\mathbb{A})$*, denoted by* $I = \oplus_{n\geq 0} I_n$*, is an ideal of* $\mathrm{H}^*(\mathbb{C})$ *consisting of nilpotents. Furthermore,* $i^*(\mathrm{H}^*(\mathbb{C}))$ *is a subring of* $\mathrm{H}^*(\mathbb{A})$*.*

Proof. The first observation is from the long exact sequence induced by

$$0 \to \mathbb{A} \xrightarrow{i} \mathbb{C} \xrightarrow{\pi} \mathbb{C}/\mathbb{A} \to 0.$$

The long exact sequence looks like

$$\cdots \to \mathrm{H}^{n-1}(\mathbb{A}) \to \mathrm{H}^n(\mathbb{C}/\mathbb{A}) \xrightarrow{\pi^*} \mathrm{H}^n(\mathbb{C}) \xrightarrow{i^*} \mathrm{H}^n(\mathbb{A}) \to \mathrm{H}^{n+1}(\mathbb{C}/\mathbb{A}) \to \cdots.$$

It's a canonical result that i^* and π^* induce two ring homomorphisms, still denoted by i^* and π^*. We get the image $I = \pi^*(\mathrm{H}^*(\mathbb{C}/\mathbb{A})) \subset \oplus_{n\geq 1} \mathrm{H}^n(\mathbb{C})$ because of $H^0(\mathbb{C}/\mathbb{A}) = 0$. Now we show I is an ideal. Consider an arbitrary cocycle $f \in C_n^*(\mathbb{C}/\mathbb{A})$. For any cocycle $g \in C_m^*(\mathbb{C})$, we get a cocycle $(\pi^*(f) \cup g) \in C_{n+m}^*(\mathbb{C})$. Let's compute $(\pi^*(f) \cup g)(\sigma)$ for any base element $\sigma \in C_{n+m}$. By definition of cup product, we have

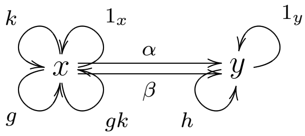
$$(\pi^*(f)\cup g)(\sigma) = \pi^*(f)(\sigma|_{(x_0,\cdots,x_n)})\cdot g(\sigma|_{(x_n,\cdots,x_{n+m})}) = f(\pi(\sigma|_{(x_0,\cdots,x_n)}))\cdot g(\sigma|_{(x_n,\cdots,x_{n+m})}).$$

It's zero when σ only consists of automorphisms. So $\pi^*(f) \cup g$ is actually of the form $\sum_{\sigma\in\Gamma} r_\sigma f_\sigma$, where Γ is a set of chains of length $n + m$ containing at least one non-isomorphism. This element $h = \sum_{\sigma\in\Gamma} r_\sigma f_\sigma$ can be regarded as a cocycle in $C^*_{n+m}(\mathbb{C}/\mathbb{A})$, and now $\pi^*(h) = \pi^*(f) \cup g$. So I is an ideal of the ring $\mathrm{H}^*(\mathbb{C})$. Elements of I are nilpotent because the category $\mathcal{C}$ is finite. If k is the maximum length of chains of non-isomorphisms in $\mathcal{C}$, then for any $f \in I$, we must have $f^{k+1} = 0$ by the definition of cup product.

As for the image $i^*(\mathrm{H}^*(\mathbb{C}))$, it contains the identity $\sum_{x\in\mathrm{Ob}\,\mathcal{C}} f_x$. So it's a subring of $\mathrm{H}^*(\mathbb{A})$. In fact, it's isomorphic to $\mathrm{H}^*(\mathbb{C})/I$ because of the exactness of the long exact sequence. □

6.3 The existence of a cohomology ring with non-nilpotent elements

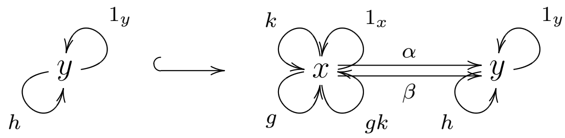
It's necessary to show that there exist non-nilpotent elements in the cohomology ring of at least one EI-category, because otherwise there is no need to make the above conjecture. One of the examples is the following category $\mathcal{C}$:



,

where $\alpha k = \alpha, \alpha g = \alpha(gk) = \beta, \beta k = \beta, \beta g = \beta(gk) = \alpha$ and $h\alpha = \beta, h\beta = \alpha$.

There is a full subcategory $\mathcal{D}$ of $\mathcal{C}$, which is equivalent to the cyclic group of order 2, on the left of the following diagram



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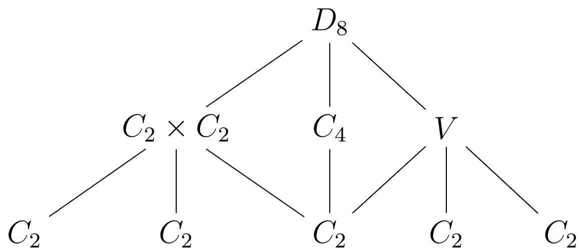
Using Quillen's theorem A we can show these two categories are homotopy equivalent. Therefore the cohomology ring of $\mathcal{C}$ is isomorphic to that of $\mathcal{D}$, which is well-known to contain non-nilpotent elements.

Chapter 7

An example

In this chapter, we plan to use an example to illustrate our main results throughout this thesis. From now on, C_n denotes a cyclic group of order n.

Let's take the symmetric group on four letters S_4. There are three Sylow 2-subgroups of S_4, which are isomorphic to D_8 (the dihedral group of order 8) and contain the same subgroup V of order 4, where $V = \{(), (12)(34), (13)(24), (14)(23)\}$. In fact, it's not hard to see that $S_4 \cong V \rtimes S_3$. Below is a poset $\mathcal{C}$ which consists of the non-trivial 2-subgroups of a Sylow 2-subgroup, denoted as D_8, from which one can get the poset of all (non-trivial) 2-subgroups of S_4.



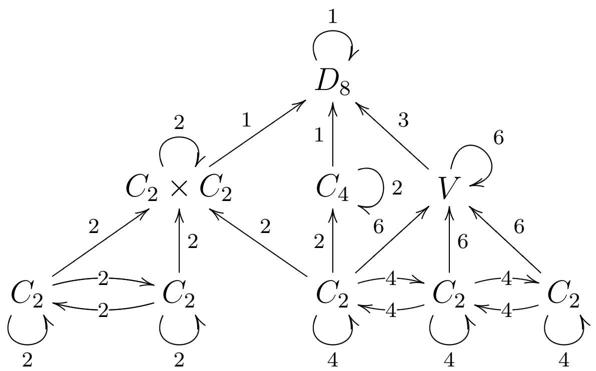
In this particular poset, we have the following groups:

$D_8 = \{(), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$,

$V = \{(), (12)(34), (13)(24), (14)(23)\}$,

$C_4 = \{(), (1324), (12)(34), (1423)\}$, $C_2 \times C_2 = \{(), (12), (34), (12)(34)\}$, and the five C_2's: $\{(), (12)\}$, $\{(), (34)\}$, $\{(), (12)(34)\}$, $\{(), (13)(24)\}$ and $\{(), (14)(23)\}$. Note that the center of this D_8 is $\{(), (12)(34)\}$.

Let $O_2(S_4)$ be the 2-orbit category of S_4. This is a category whose objects are the non-trivial 2-subgroups of S_4. If H, K are two 2-subgroups of S_4, then the set of morphisms from H to K is defined as $\mathrm{Hom}(H,K) = K\backslash N_{S_4}(H,K)$, where $N_{S_4}(H,K)$ is called the transporter, consisting of elements $g \in S_4$ such that ${}^gH \subset K$. It's easy to see that the 2-orbit category $O_2(S_4)$ is equivalent to the the full subcategory whose objects are the 2-subgroups of a fixed Sylow 2-subgroup of S_4, because every 2-subgroup is isomorphic in $O_2(S_4)$ to a subgroup of a fixed Sylow 2-subgroup. The following category is such a full subcategory of $O_2(S_4)$, and as an abuse of notations, we'll still name it $O_2(S_4)$.



Here the numbers denote the sizes of the sets of irreducible morphisms between two adjacent objects, or the sizes of endomorphisms groups. Note that there are some other non-isomorphisms from the C_2's to D_8. For instance, there is one (reducible) morphism from $\{(), (12)\}$ to D_8 and three (reducible) morphisms from $\{(), (12)(34)\}$ to D_8. A good way to calculate the morphisms in such an orbit category is to use the powerful computer software GAP. Let's list some of the morphism sets (by representatives):

$\mathrm{Hom}(C_2 \times C_2, D_8) = \{()\}$, $\mathrm{Hom}(\langle(12)\rangle, C_2 \times C_2) = \{(), (13)(24)\}$,

$\mathrm{Hom}(\langle(12)(34)\rangle, C_2 \times C_2) = \{(), (13)(24)\}$, $\mathrm{Hom}(\langle(12)(34)\rangle, C_4) = \{(), (12)\}$,

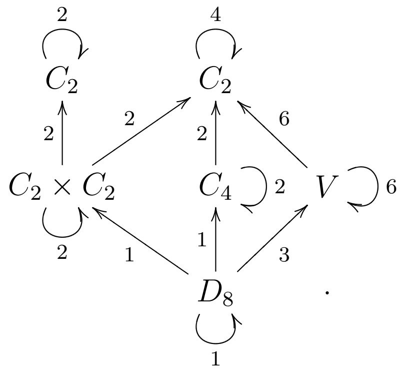
$\mathrm{Hom}(\langle(12)(34)\rangle, V) = \{(), (12), (13), (14), (123), (132)\}$,

$\mathrm{Hom}(V, D_8) = \{(), (123), (132)\}$.

By direct computation, we can also completely understand the actions of the automorphism groups $\mathrm{Aut}(x)$ and $\mathrm{Aut}(y)$ on $\mathrm{Hom}(x,y)$, for any $x, y \in \mathrm{Ob}\, O_{\mathcal{C}}$. For future reference, we describe some of the actions. Note that all morphisms in such an orbit category are epimorphisms, and this means that $\alpha\beta = \alpha'\beta$ implies $\alpha = \alpha'$. We can find, for instance, $\mathrm{Aut}(V)$ acts transitively and freely on $\mathrm{Hom}(\langle(12)(34)\rangle, V)$

and $\mathrm{Aut}(C_4)$ acts transitively and freely on $\mathrm{Hom}(\langle(12)(34)\rangle, C_4)$. Here $\mathrm{Aut}(V) \cong S_3$ is presented by the set of representatives $\{(), (12), (13), (14), (123), (132)\}$.

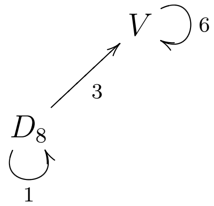
In practice, people care about the contravariant functors from $O_2(S_4)$ to R-mod. While in order to be consistent with our conventions, we consider the covariant functors from the opposite category $O_2^{op}(S_4)$ to R-mod. For simplicity, we use the skeleton of $O_2^{op}(S_4)$, instead of $O_2^{op}(S_4)$ itself, to illustrate our calculations. Thus the category $\mathcal{O}$ to be studied is as follows



We're going to give the following information for this category $\mathcal{O}$ (for $R = \mathbb{C}$ or $\mathbb{F}_2$):

1. **The vertex $\mathcal{V}_{\underline{R}}$**

   The vertex of $\underline{R}$ is the following full subcategory:



   This category $\mathcal{V}_{\underline{R}}$ is R-co-taut in $\mathcal{O}$. So we have $\varprojlim^*_{\mathcal{O}} M \cong \varprojlim^*_{\mathcal{V}_{\underline{R}}} \mathrm{Res}_\iota M$ by Lemma 4.4.22. In order to produce the minimal projective resolution for the trivial $R\mathcal{V}_{\underline{R}}$-module, we first give the structure of $R\mathcal{V}_{\underline{R}}$ as follows. First of all we have $R\mathcal{V}_{\underline{R}} = R\mathcal{V}_{\underline{R}} \cdot 1_{D_8} \oplus R\mathcal{V}_{\underline{R}} \cdot 1_V$. When $R = \mathbb{C}$, the two projective modules can be described by

$$\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array} \quad \text{and} \quad 1 \oplus -1 \oplus 2 \oplus 2,$$

or $\mathbb{C}\mathcal{V}_{\underline{\mathbb{C}}} = P_{D_8,1}\oplus[P_{V,1}\oplus P_{V,-1}\oplus P_{V,2}^2]$. When $R = \mathbb{F}_2$, $\mathbb{F}_2\mathcal{V}_{\underline{\mathbb{F}_2}} = P_{D_8,1}\oplus[P_{V,1}\oplus P_{V,2}^2]$, or we can describe the two projectives as

$$\begin{array}{ccc} & 1 & \\ 1 & & 2 \end{array} \quad \text{and} \quad \begin{array}{c} 1 \\ 1 \end{array} \oplus 2 \oplus 2.$$

Now for either choice of the base ring R, the trivial $R\mathcal{V}_{\underline{R}}$-module $\underline{R}$ has a (minimal) projective resolution of length 2: $0 \to P_{V,2} \to P_{D_8,1} \to \underline{R} \to 0$, where $P_{D_8,1} = R\mathcal{V}_{\underline{R}} \cdot 1_{D_8}$ is the projective cover of $\underline{R}$, and $P_{V,2} = S_{V,2}$ is the projective simple of $R\,\mathrm{Aut}(V) \cong RS_3$ of dimension 2, regarded as an $R\mathcal{V}_{\underline{R}}$-module. Thus we have $\varprojlim^i_{\mathcal{O}} M \cong \varprojlim^i_{\mathcal{V}_{\underline{R}}} M = 0$ for any $M \in R\mathcal{C}$-mod if $i > 1$. One can compare these calculations with a formula of Jackowski-Słomińska [25] (quoted here as Corollary 3.3.4). Fixing a finite group G and a prime p, a subgroup H is p-radical if the group $N_G(H)/H$ has no normal p-subgroups. Their formula asserts that the p-orbit category of a group G, $O_p(G)$, can always be replaced by its full subcategory consisting of just p-radical subgroups when one wants to compute higher limits over $O_p^{op}(G)$. Note that in our example the two subgroups of S_4, D_8 and V, are the only 2-radical subgroups of S_4 which are contained in D_8. Furthermore Grodal [17] found that the higher limits over the full subcategory of $O_p^{op}(G)$, which consists of all non-trivial p-radical subgroups, vanish if the degree is bigger than the length of the underlying poset of the subcategory. In our example, the 2-orbit category $O_2^{op}(G)$ is replaced by its full subcategory $\mathcal{V}_{\underline{R}}$, which consists of the two 2-radical subgroups D_8, V. Since the underlying poset of $\mathcal{V}_{\underline{R}}$ has length one, we must have $\varprojlim^i_{\mathcal{V}_{\underline{R}}} M = 0$ if $i > 1$, and this verifies our previous calculations.

Remark 7.0.1. *If we consider the 2-orbit category of D_8, instead of S_4, then the only 2-radical group is D_8 itself. It's easy to see that D_8 becomes the initial object of $O_2^{op}(D_8)$, and $\{D_8\}$ is the vertex of $\underline{R}$.*

Remark 7.0.2. *We can check that the above category $\mathcal{V}_{\underline{R}}$ is isomorphic to the opposite of the 2-orbit category of S_3 which has objects C_2 and $\{()\}$. Thus*

$$\mathrm{H}^*(O_2^{op}(S_4), M) \cong \mathrm{H}^*(O_2^{op}(S_3), M),$$

for any $M \in RO_2^{op}(S_4)$*-mod. If we recall Jackowski-McClure-Oliver's notation [24]*

$$\Lambda^*(K, M) = \mathrm{H}^*(O_p^{op}(K), M(\{()\})),$$

where M *is an atomic functor concentrated on the trivial subgroup* $\{()\}$ *of a group* K*, then we can compare our formula for computing* $\varprojlim^*_{O_2^{op}(S_4)} M$ *with theirs*

$$\mathrm{H}^*(O_p^{op}(G), M) \cong \Lambda^*(N_G(P)/P, M(\{()\})),$$

where M *is an atomic functor concentrated on the conjugacy class of the subgroup* P*. Indeed, if we take* $N = S_4, P = V$ *and* $p = 2$ *in their formula, then their formula implies* $\mathrm{H}^*(O_2^{op}(S_4), M) \cong \mathrm{H}^*(O_2^{op}(S_3), M)$ *for any atomic* M *that is concentrated on* V*, since* $N_{S_4}(V)/V \cong S_3$*.*

2. **Vertices for other modules and Green correspondence**

According to Proposition 4.4.19, the only objects of the vertex of each indecomposable projective module P are the P-minimal objects. If M is an indecomposable atomic module (Definition 3.3.3), concentrated on $[x]$, then $\mathcal{V}_M$ is the full subcategory consisting of its support $[x]$ and the objects y such that $\mathrm{Irr}(x, y) \neq \emptyset$ (Proposition 4.4.20). Simple modules are atomic. Thus the simple module $S_{D_8,1}$ has as its vertex the full subcategory of $\mathcal{O}$ consists of objects $D_8, C_2 \times C_2, C_4$ and V. The simple $\mathbb{C}\mathcal{O}$-modules $S_{V,1}, S_{V,-1}$ (if $R = \mathbb{C}$) and $S_{V,2}$ all have the full subcategory with two objects, V and C_2, as their vertex.

The object $C_2 = \{(), (12)(34)\}$ has an automorphism of order 4, which is isomorphic to $C_2 \times C_2$. Since there are infinitely many indecomposable $\mathbb{F}_2\mathcal{O}$-modules whose vertex is $\{C_2\}$, $\mathbb{F}_2\mathcal{O}$ is of infinite representation type because there exists a Green correspondence (Corollary 4.4.8). More generally, if an EI-category $\mathcal{E}$ has an object x whose automorphism group algebra $R\,\mathrm{Aut}(x)$ is of infinite representation type, then so is $R\mathcal{E}$ because, by Theorem 4.4.4, each indecomposable $R\,\mathrm{Aut}(x)$-module gives rise to an indecomposable $R\mathcal{E}$-module by tensor induction, and any two modules obtained in this way are not isomorphic to each other since their evaluations at x are not.

3. **The groups** $\mathrm{Ext}^*_{R\mathcal{O}}(S_1, S_2)$ **and** $\varprojlim^*_{\mathcal{O}} M$ **etc.**

In this example, if there exists only one reducible morphism from the support of S_1 to that of S_2, then $\mathrm{Ext}^0_{R\mathcal{O}}(S_1, S_2) = \mathrm{Ext}^1_{R\mathcal{O}}(S_1, S_2) = 0$ (Proposition 5.1.5). For S_1 and S_2 that share the same support, the Ext groups are isomorphic to those over the automorphism group of their support while S_1 and S_2 are regraded as group modules (Corollary 5.1.4).

We compute two examples for which there exists at least one irreducible non-isomorphism between the supports of the two simples. First of all, if we take the supports to be D_8 and V, then there are three different kinds of groups that we need to consider: $\mathrm{Ext}^*_{R\mathcal{O}}(S_{D_8,1}, S_{V,1})$, $\mathrm{Ext}^*_{R\mathcal{O}}(S_{D_8,1}, S_{V,2})$ and $\mathrm{Ext}^*_{R\mathcal{O}}(S_{D_8,1}, S_{V,-1})$ when $R = \mathbb{C}$. Note that we can restrict our attention to the full subcategory consisting of only two objects D_8 and V (Theorem 5.1.3). Within this full subcategory we can write out the minimal resolution of $S_{D_8,1}$ for each situation. Then the Ext groups are isomorphic to the Hom's, because $S_{V,?}$ and $S_{V,2}$ are simple.

(a) When $R = \mathbb{C}$, the minimal resolution is

$$0 \to P_{V,2} \oplus P_{V,1} \to P_{D_8,1} \to S_{D_8,1} \to 0,$$

since the $\mathbb{C}$-span of $\mathrm{Hom}_{\mathcal{O}}(D_8, V) = \{(), (123), (132)\}$, as a $\mathbb{C}\,\mathrm{Aut}(V)(\cong \mathbb{C}S_3)$-module, can be decomposed into a direct sum of two projective simples

$$\mathbb{C}\{() + (123) + (132)\} \oplus \mathbb{C}\{(123) - (), (132) - ()\}.$$

Thus we have

$$\mathrm{Ext}^0_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,1}) \cong \mathrm{Hom}(P_{D_8,1}, S_{V,1}) = 0,$$

$$\mathrm{Ext}^0_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,-1}) \cong \mathrm{Hom}(P_{D_8,1}, S_{V,-1}) = 0,$$

$$\mathrm{Ext}^0_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,2}) \cong \mathrm{Hom}(P_{D_8,1}, S_{V,2}) = 0,$$

$$\mathrm{Ext}^1_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,1}) \cong \mathrm{Hom}(P_{V,1} \oplus P_{V,2}, S_{V,1}) \cong \mathbb{C},$$

$$\mathrm{Ext}^1_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,-1}) \cong \mathrm{Hom}(P_{V,1} \oplus P_{V,2}, S_{V,-1}) = 0,$$

$$\mathrm{Ext}^1_{\mathbb{C}\mathcal{O}}(S_{D_8,1}, S_{V,2}) \cong \mathrm{Hom}(P_{V,1} \oplus P_{V,2}, S_{V,2}) \cong \mathbb{C}$$

and $\mathrm{Ext}^i_{\mathbb{C}\mathcal{O}}(-,-) = 0$ for $i \geq 2$;

(b) When $R = \mathbb{F}_2$, the minimal resolution is

$$\cdots \to P_{V,1} \to P_{V,1} \to P_{V,1} \oplus P_{V,2} \to P_{D_8,1} \to S_{D_8,1} \to 0,$$

which is infinite with all places filled in by $P_{V,1}$ except at degrees 0 and 1. It is so because the $\mathbb{F}_2$-span of $\mathrm{Hom}(D_8, V) = \{(), (123), (132)\}$ still decomposes into a direct sum of the trivial module and the dimension 2 projective simple of $\mathbb{F}_2 S_3$, but the trivial module is no longer projective which has a projective cover of dimension 2 whose socle (= radical) is isomorphic to $\mathbb{F}_2$. Thus we use the property of minimal resolutions to get

$$\mathrm{Ext}^0_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,1}) \cong \mathrm{Hom}(P_{D_8,1}, S_{V,1}) = 0,$$

$$\mathrm{Ext}^0_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,2}) \cong \mathrm{Hom}(P_{D_8,1}, S_{V,2}) = 0,$$

$$\mathrm{Ext}^1_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,1}) \cong \mathrm{Hom}(P_{V,1} \oplus P_{V,2}, S_{V,1}) \cong \mathbb{F}_2,$$

$$\mathrm{Ext}^1_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,2}) \cong \mathrm{Hom}(P_{V,1} \oplus P_{V,2}, S_{V,2}) \cong \mathbb{F}_2,$$

$$\mathrm{Ext}^i_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,1}) \cong \mathrm{Hom}(P_{V,1}, S_{V,1}) \cong \mathbb{F}_2$$

and $\mathrm{Ext}^i_{\mathbb{F}_2\mathcal{O}}(S_{D_8,1}, S_{V,2}) \cong \mathrm{Hom}(P_{V,1}, S_{V,2}) \cong 0$ for $i \geq 2$.

If we want to know $\varprojlim^*_{\mathcal{O}} S_{D_8,1}$ and $\varprojlim^*_{\mathcal{O}} S_{V,?}$, then we can first apply the formula $\varprojlim^*_{\mathcal{O}} M \cong \varprojlim^*_{\mathcal{V}_{\underline{R}}} \mathrm{Res}_\iota M$ so that these groups become $\varprojlim^*_{\mathcal{V}_{\underline{R}}} S_{D_8,1}$ and $\varprojlim_{\mathcal{V}_{\underline{R}}} S_{V,?}$. Using the minimal projective resolution of $\underline{R}$, we can get that all these groups vanish except that $\varprojlim_{\mathcal{O}} S_{D_8,1} \cong \varprojlim_{\mathcal{V}_{\underline{R}}} S_{D_8,1} \cong R$ and $\varprojlim^1_{\mathcal{O}} S_{V,2} \cong \varprojlim^1_{\mathcal{V}_{\underline{R}}} S_{V,2} \cong R$.

4. The cohomology ring

In part 1, we obtained a projective resolution of the $R\mathcal{V}_{\underline{R}}$-module $\underline{R}$, and it can be used to calculate the cohomology ring $\mathrm{Ext}^*_{R\mathcal{O}}(\underline{R}, \underline{R})$. By Proposition 4.4.25, this cohomology ring is isomorphic to $\mathrm{Ext}^*_{R\mathcal{V}_{\underline{R}}}(\underline{R}, \underline{R})$ for the full subcategory $\mathcal{V}_{\underline{R}}$. Now from the projective resolution of $R\mathcal{V}_{\underline{R}}$-module $\underline{R}$,

$$0 \to P_{V,2} \to P_{D_8,1} \to \underline{R} \to 0,$$

we get a cochain complex $0 \to \mathrm{Hom}(P_{D_8,1}, \underline{R}) \to \mathrm{Hom}(P_{V,2}, \underline{R}) \to 0$. Since $\mathrm{Hom}_{\mathcal{V}_{\underline{R}}}(P_{V,2}, \underline{R}) \cong \mathrm{Hom}_{RS_3}(S_{V,2}, R) = 0$, we have $\mathrm{Ext}^1_{R\mathcal{V}_{\underline{R}}}(\underline{R}, \underline{R}) \cong \mathrm{Hom}(P_{V,2}, \underline{R}) \cong$

0, and hence $\mathrm{Ext}^0_{R\mathcal{V}_{\underline{R}}}(\underline{R}, \underline{R}) \cong \mathrm{Hom}(P_{D_8,1}, \underline{R})$ which is isomorphic to R. Of course we can tell $\mathrm{Ext}^0_{R\mathcal{V}_{\underline{R}}}(\underline{R}, \underline{R}) \cong R$ directly from the fact that $\mathcal{V}_{\underline{R}}$ is connected.

The cohomology ring of $\mathcal{O}$, $\mathrm{Ext}^*_{R\mathcal{O}}(\underline{R}, \underline{R})$, is isomorphic to R for $R = \mathbb{C}$ or $R = \mathbb{F}_2$.

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