Face Numbers of Poset Associahedra: Results and Conjectures

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Abstract

For any finite connected poset P, Galashin introduced a simple convex $(|P|-2)$-dimensional polytope $\mathscr{A}(P)$ called the poset associahedron. We study the face numbers of poset associahedra. First, we show that the face numbers of $\mathscr{A}(P)$ only depend on the comparability graph of P. Second, for a certain family of posets, whose poset associahedra interpolate between the classical permutohedron and associahedron, we give a simple combinatorial interpretation of the h-vector, which allows us to prove real-rootedness of some of their h-polynomials. Then, we conjecture a general identity involving the h-polynomials. Finally, we survey some results and conjectures on the γ-positivity of poset associahedra.

Contents

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1 Introduction

For a finite connected poset P, Galashin introduced the *poset associahedron* $\mathscr{A}(P)$ (see [Gal21]). The faces of $\mathscr{A}(P)$ correspond to *tubings* of P, and the vertices of $\mathscr{A}(P)$ correspond to *maximal tubings* of P; see Section 2.3 for the definitions. $\mathscr{A}(P)$ can also be described as a compactification of the configuration space of order-preserving maps $P \to \mathbb{R}$.

Many polytopes can be described as poset associahedra, including permutohedra and associahedra. In particular, when P is the claw poset, i.e. P consists of a unique minimal element 0 and n pairwise-incomparable elements, then $\mathscr{A}(P)$ is the n-permutohedron. On the other hand, when P is a chain of $n+1$ elements, i.e. $P = C_{n+1}$, then $\mathscr{A}(P)$ is the associahedron K_{n+1}.

In this thesis, we survey recent results and conjectures concerning the face numbers of poset associahedra. Most of the results here are joint work with Andrew Sack (see [NS23a, NS23b]).

For a d-dimensional polytope P, the f-vector of P is the sequence $(f_0(P), \dots, f_d(P))$ where $f_i(P)$ is the number of i-dimensional faces of P. The f-polynomial of P is

$$f_P(t) = \sum_{i=0}^{d} f_i(P) t^i.$$

For simple polytopes such as poset associahedra, it is often better to consider the smaller and still nonnegative h-vector and h-polynomial defined by the relation

$$f_P(t) = h_P(t+1).$$

It is well-known that when P is a simple polytope, its h-vector is symmetric: $h_i(P) = h_{d-i}(P)$. Thus, there is the even smaller *γ-vector* and *γ-polynomial* defined by

$$h_P(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(P) t^i (1+t)^{d-2i} = (1+t)^d \gamma_P\left(\frac{t}{(1+t)^2}\right).$$

1.1 Comparability invariance

The *comparability graph* of a poset P is a graph $C(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of P is said to be *comparability invariant* if it only depends on $C(P)$. We will show in Section 3 that the face numbers of poset associahedra is a comparability invariant.

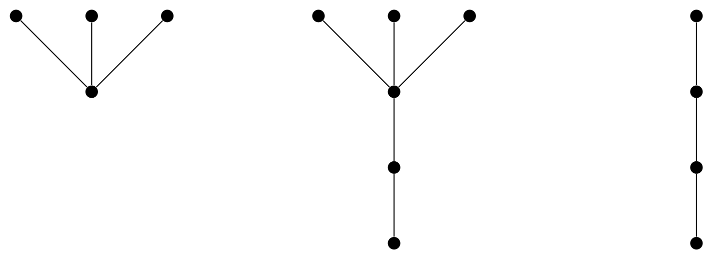
Theorem 3.1 ([NS23b, Theorem 1.1])**.** *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

1.2 Stack-sorting

Stack-sorting is a function $s : \mathfrak{S}_n \to \mathfrak{S}_n$ which attempts to sort the permutations w in $\mathfrak{S}_n$ in linear time, not always sorting them completely (see definition in Section 4.3). A permutation $w \in \mathfrak{S}_n$ is stack-sortable if $s(w) = 12\dots n$. It is well-known that stack-sortable permutations are exactly 231-avoiding permutations. Thus, we have an alternative interpretation for the h-vector of $\mathscr{A}(C_{n+1})$: h_i counts the number of permutations in $s^{-1}(12\dots n)$ with exactly i descents.

In Section 4, we will study broom posets $A_{n,k} = C_{n+1} \oplus A_k$ where A_k is the antichain of k elements. In particular, $A_{0,k}$ is a claw poset, and $A_{n,0}$ is the chain C_{n+1}. For example,

Figure 1 shows three broom posets. The left poset $A_{0,3}$ is a claw poset, the right poset $A_{3,0}$ is a chain, and the middle poset is an intermediate broom poset.



$A_{0,3}$ $\qquad$ $A_{2,3}$ $\qquad$ $A_{3,0}$

Figure 1: Some broom posets

Surprisingly, the h-vector of $\mathscr{A}(A_{n,k})$ is also counted by descents of stack-sorting preimages. Let $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$, we prove the following generalization of the above classic result.

Theorem 4.1 ([NS23a, Theorem 4.8])**.** *Let $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i$ for all $i > k\}$ and $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

In the process of proving Theorem 4.1, we find the size of $s^{-1}(\mathfrak{S}_{n,k})$ in terms of $k!$ and the Catalan convolution $C_n^{(k)}$, which will be introduced in Section 4.2.

Corollary 4.14. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Note that $C_n^{(0)}$ is the classic Catalan number C_n. Thus, setting $k = 0$ in Corollary 4.14, we obtain the classic result that $s^{-1}(12\ldots n) = C_n$. Finally, in Section 4.5, we will use a "happy coincidence" in stack-sorting to prove real-rootedness of the h-polynomials of $\mathscr{A}(A_{n,2})$.

Theorem 4.33. *Let $H_n(x)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(x)$ is real-rooted.*

1.3 An h-vector identity

In Section 5, we will make a conjecture that generalizes the happy coincidence above. We say S is an *autonomous subset* of a poset P if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

In other words, every element in $P - S$ "sees" every element in S the same.

In Section 5.1, we will introduce three families of polynomials $B_w(x), G_w(x), \tilde{F}_w(x)$ in $\mathbb{Z}[x]$, indexed by permutations w in $\mathfrak{S}_n$. Then, we make the following conjectures.

Conjecture 5.1. *Let P be a poset with a proper autonomous subposet S that is a chain of size n. For $1 \leq i \leq n$, let P_i be the poset obtained from P by replacing S by an antichain of size i. Let $h_P(x)$, $h_{P_1}(x)$, …, $h_{P_n}(x)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, …, $\mathscr{A}(P_n)$, respectively. Then,*

$$h_P(x) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} B_w(x) h_{P_{\ell(w)}}(x).$$

We will show that Conjecture 5.1 follows from the following conjecture.

Conjecture 5.2. *For all n,*

$$\sum_{w\in\mathfrak{S}_n} t^{\ell_w} G_w(x) = \sum_{w\in\mathfrak{S}_n} t(t+x)\dots(t+(\ell_w-1)x)\tilde{F}_w(x).$$

1.4 γ-positivity

In Section 6, we give combinatorial interpretations of the h-and-γ-vectors of cyclic fence posets in terms of colored balanced paths (see definitions in Section 6.1). For example, Figure 2 gives examples of two cyclic fence posets: CF_6 and CF_7.

(a) CF_6 (b) CF_7

Figure 2: Cyclic fence posets

Theorem 6.5. *For $CF_{2(n+1)}$, the h-vector is given by*

$$h_i = |\{w \in CP_{n,n} \mid \#\textit{red peak steps} - \#\textit{blue peak steps} = 2(i-n)\}|.$$

For CF_{2n+1}, the h-vector is given by

$$h_i = |\{w \in CP_{n-1,n} \mid \#\textit{red side steps} - \#\textit{blue side steps} = 2(i-n)+1\}|.$$

Corollary 6.8. *For $CF_{2(n+1)}$, the γ-vector is given by*

$$\gamma_i = 4^i \binom{n}{i}^2.$$

For CF_{2n+1}, the γ-vector is given by

$$\gamma_i = 4^i \binom{n}{i}\binom{n-1}{i}.$$

In particular, the poset associahedra of cyclic fence posets are γ-positive.

From Corollary 6.8 and computational evidence, we make a few conjectures about γ-positivity of poset associahedra.

Conjecture 6.9. *All poset associahedra are γ-positive.*

Conjecture 6.10. *Let P be a connected poset on n elements, and P is not C_n or $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$. Then the h-and-γ-vectors of $\mathscr{A}(P)$ satisfy*

$$h_{C_n} \lll h_P \lll h_{K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}},$$

$$\gamma_{C_n} \lll \gamma_P \lll \gamma_{K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}}.$$

Here, we say $(a_1,\dots,a_k) \lll (b_1,\dots,b_k)$ if $a_i < b_i$ for all $1 \le i \le k$. Conjecture 6.9 can be made even stronger as follows.

Conjecture 6.11. *The h-polynomials of poset associahedra are real-rooted.*

2 Preliminaries

2.1 Polynomials and sequences

A polynomial $p(x) = a_0 + a_1x + \ldots + a_nx^n \in \mathbb{R}_{\geq 0}[x]$ is called

- *symmetric* if $a_i = a_{n-i}$ for all $i \in [0, n]$;
- *unimodal* if $a_0 \leq a_1 \leq \ldots \leq a_j \geq a_{j+1} \geq \ldots \geq a_n$ for some j;
- *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for all $i \in [n-1]$;
- *real-rooted* if all complex roots of $p(x)$ are real.

When $p(x)$ is symmetric, it has a unique expansion in terms of binomials $t^i(1+t)^{n-2i}$ for $0 \leq i \leq n/2$, i.e. we can write

$$p(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}.$$

We say $p(x)$ is *γ-nonnegative* (resp. *γ-positive*) if all coefficients γ_i in the above expansion are nonnegative (resp. positive). Since the f-and-h-vectors are nonnegative and have no internal zeros, we have the following implications among these properties:

- real-rooted $\Rightarrow$ log-concave $\Rightarrow$ unimodal;
- symmetric and real-rooted $\Rightarrow$ γ-nonnegative $\Rightarrow$ symmetric and unimodal.

Finally, we say a sequence $(a_0, a_1, \ldots, a_n)$ has property X if its generating function $p(x) = a_0 + a_1x + \ldots + a_nx^n$ has property X.

2.2 Polytope and face numbers

A *convex polytope* P is the convex hull of a finite collection of points in $\mathbb{R}^n$. The *dimension* of a polytope is the dimension of its affine span. A face F of a convex polytope P is the set of points in P where some linear functional achieves its maximum on P. Faces that consist of a single point are called *vertices* and 1-dimensional faces are called *edges* of P. A d-dimensional polytope P is *simple* if any vertex of P is incident to exactly d edges.

For a d-dimensional polytope P, the *face number* $f_i(P)$ is the number of i-dimensional faces of P. In particular, $f_0(P)$ counts the vertices and $f_1(P)$ counts the edges of P. The sequence $(f_0(P), f_1(P), \ldots, f_d(P))$ is called the *f-vector* of P, and the polynomial

$$f_P(t) = \sum_{i=0}^{d} f_i(P)t^i$$

is called the *f-polynomial* of P. The *h-vector* $(h_0(P), \ldots, h_d(P))$ and *h-polynomial* $h_P(t) = \sum_{i=0}^{d} h_i(P)t^i$ are defined by the relation

$$f_P(t) = h_P(t+1).$$

It is well-known that when P is a simple polytope, its h-vector is nonnegative and satisfies the Dehn-Sommerville symmetry: $h_i(P) = h_{d-i}(P)$. When the h-polynomial is symmetric, recall that it has a unique expansion in terms of (centered) binomials $t^i(1+t)^{d-2i}$

for $0 \leq i \leq d/2$. This unique expansion gives the *γ-vector* $(\gamma_0(P), \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor}(P))$ and *γ-polynomial* $\gamma_P(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(P) t^i$ defined by

$$h_P(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i(P) t^i (1+t)^{d-2i} = (1+t)^d \gamma_P\left(\frac{t}{(1+t)^2}\right).$$

Note that the γ-vector may not be nonnegative.

2.3 Poset and poset associahedra

Definition 2.1. A *partially ordered set (poset)* is a set P with a partial order $\preceq$ satisfying the following conditions.

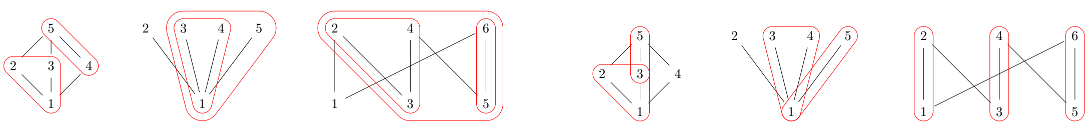
1. *Reflexivity*: $x \preceq x$ for all $x \in P$.
2. *Antisymmetry*: if $x \preceq y$ and $y \preceq x$ then $x = y$.
3. *Transitivity*: if $x \preceq y$ and $y \preceq z$ then $x \preceq z$.

We have some poset terminologies.

Definition 2.2. Let $(P, \preceq)$ be a finite poset, and $\tau, \sigma \subseteq P$ be subposets.

- τ is *connected* if it is connected as an induced subgraph of the Hasse diagram of P.
- τ is *convex* if whenever $x, z \in \tau$ and $y \in P$ such that $x \preceq y \preceq z$, then $y \in \tau$.
- τ is a *tube* of P if it is connected and convex. τ is a *proper tube* if $1 < |\tau| < |P|$.
- τ and σ are *nested* if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$. τ and σ are *disjoint* if $\tau \cap \sigma = \emptyset$.
- We say $\sigma \prec \tau$ if $\sigma \cap \tau = \emptyset$, and there exists $x \in \sigma$ and $y \in \tau$ such that $x \preceq y$.
- A *tubing* T of P is a set of proper tubes such that any pair of tubes in T is either nested or disjoint, and there is no subset $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$. We will refer to the latter condition as the *acyclic condition*.
- A tubing T is *maximal* if it is maximal under inclusion, i.e. T is not a proper subset of any other tubing.

Example 2.3. Figure 3 shows examples and non-examples of tubings of posets. Note that the right-most example in Figure 3b is a non-example since it violates the acyclic condition. In particular, if we label the tubes from right to left as τ_1, τ_2, τ_3, then we have $\tau_1 \prec \tau_2 \prec \tau_3 \prec \tau_1$.

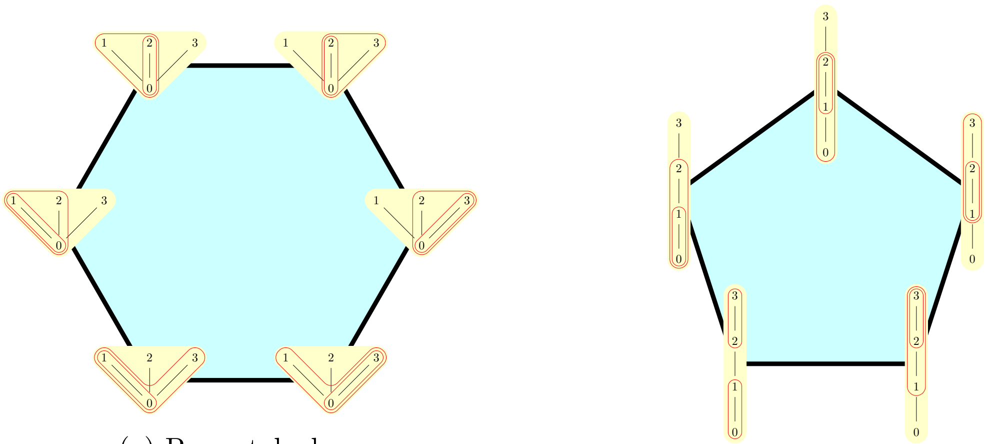


(a) Examples (b) Non-examples

Figure 3: Examples and non-examples of tubings of posets

Definition 2.4 ([Gal21, Theorem 1.2])**.** For a finite connected poset P, there exists a simple, convex polytope $\mathscr{A}(P)$ of dimension $|P|-2$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. The faces of $\mathscr{A}(P)$ correspond to tubings of P, and the vertices of $\mathscr{A}(P)$ correspond to maximal tubings of P. This polytope is called the **poset associahedron** of P.

Example 2.5. Examples of poset associahedra can be seen in Figure 4. In particular, if P is a claw, i.e. P consists of a unique minimal element 0 and n pairwise-incomparable elements as shown in Figure 4a, $\mathscr{A}(P)$ is a permutohedron. If P is a chain, $\mathscr{A}(P)$ is an associahedron.



(a) Permutohedron

(b) Associahedron

Figure 4: Permutohedron and associahedron as poset associahedra

3 Comparability invariant

The *comparability graph* of a poset P is a graph $C(P)$ whose vertices are the elements of P and where i and j are connected by an edge if i and j are comparable. A property of P is said to be *comparability invariant* if it only depends on $C(P)$. Properties of finite posets known to be comparability invariant include the order polynomial and number of linear extensions [Sta86], the fixed point property [DPW85], and the Dushnik–Miller dimension [TMS76]. It turns out that the face numbers of poset associahedra is also a comparability invariant.

Theorem 3.1 ([NS23b, Theorem 1.1])**.** *The f-vector of $\mathscr{A}(P)$ is a comparability invariant.*

In this section, we sketch the proof of Theorem 3.1 in [NS23b].

3.1 Autonomous subposets

Definition 3.2. Let P and S be posets and let $a \in P$. The *substitution* of a for S is the poset $P(a \to S)$ on the set $(P - \{a\}) \sqcup S$ formed by replacing a with S.

More formally, $x \preceq_{P(a\to S)} y$ if and only if one of the following holds:

- $x, y \in P - \{a\}$ and $x \preceq_P y$
- $x, y \in S$ and $x \preceq_S y$

- $x \in S, y \in P - \{a\}$ and $a \preceq_P y$
- $y \in S, x \in P - \{a\}$ and $y \preceq_P a$.

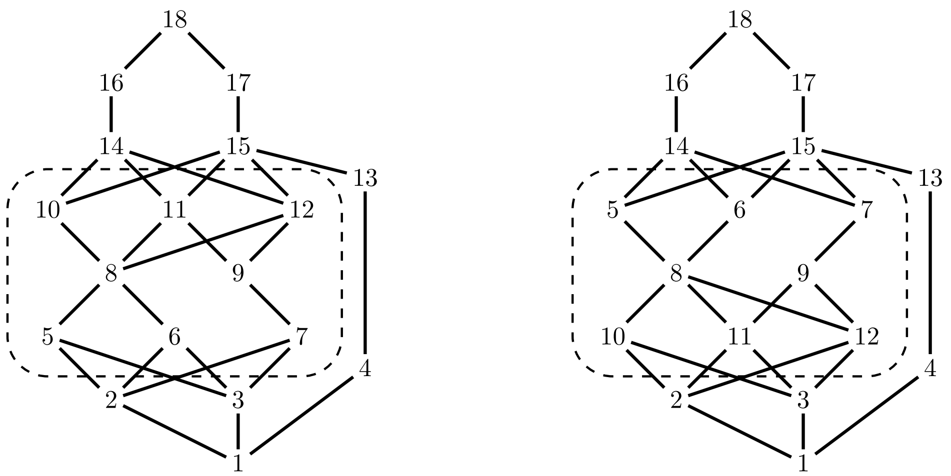
Definition 3.3. Let P be a poset and let $S \subseteq P$. S is called *autonomous* if there exists a poset Q and $a \in Q$ such that $P = Q(a \to S)$.

Equivalently, S is autonomous if for all $x, y \in S$ and $z \in P - S$, we have

$$(x \preceq z \Leftrightarrow y \preceq z) \text{ and } (z \preceq x \Leftrightarrow z \preceq y).$$

Definition 3.4. For a poset S, the *dual poset* S^{op} is defined on the same ground set where $x \preceq_S y$ if and only if $y \preceq_{S^{\mathrm{op}}} x$. The *flip* of S in $P = Q(a \to S)$ is the replacement of P by $Q(a \to S^{\mathrm{op}})$.

See Figure 5a for an example of an autonomous subset and Figure 5b for an example of a flip.



(a) An autonomous subset S of a poset P.

(b) A flip of S.

Figure 5

We have the following key lemma.

Lemma 3.5 ([DPW85, Theorem 1])**.** *If P and P' are finite posets such that $C(P) = C(P')$ then P and P' are connected by a sequence of flips of autonomous subsets.*

By Lemma 3.5, in order to prove that a property is comparability invariant, we only need to prove that it is preserved under flips.

3.2 Proof sketch of Theorem 3.1

Let $P = Q(a \to S)$ and $P' = Q(a \to S^{\mathrm{op}})$. By an abuse of notation, we let $\mathscr{A}(P)$ also refer to the set of tubings of P. Our goal is to build a bijection $\Phi_{P,S} : \mathscr{A}(P) \to \mathscr{A}(P')$ such that for any tubing $T \in \mathscr{A}(P)$, we have $|T| = |\Phi_{P,S}(T)|$. Let $T \in \mathscr{A}(P)$, we will describe how to construct $T' := \Phi_{P,S}(T)$.

Definition 3.6. A tube $\tau \in T$ is *good* if $\tau \subseteq P - S$, $\tau \subseteq S$, or $S \subseteq \tau$ and is *bad* otherwise. We denote the set of good tubes by T_{good} and the set of bad tubes by T_{bad}.

The key idea of defining $\Phi_{P,S}$ is to decompose T_{bad} into a triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$ where $\mathcal{L}$ and $\mathcal{U}$ are nested sequences of sets, some of which may be marked, contained in $P - S$ and $\mathcal{M}$ is an ordered set partition of S. We build the decomposition in such a way so that we can uniquely recover T_{bad} from $(\mathcal{L}, \mathcal{M}, \mathcal{U})$. Then, we construct T' by keeping T_{good} and replacing T_{bad} by T'_{bad}, which is obtained from $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ where $\overline{\mathcal{M}}$ is the reverse of $\mathcal{M}$. We decompose T_{bad} as follows.

Definition 3.7. A tube $\tau \in T_{\text{bad}}$ is called *lower* (resp. *upper*) if there exist $x \in \tau - S$ and $y \in \tau \cap S$ such that $x \preceq y$ (resp. $y \preceq x$). We denote the set of lower tubes by T_L and the set of upper tubes by T_U.

Lemma 3.8 (Structure Lemma)**.** *T_{bad} is the disjoint union of T_L and T_U. Furthermore, T_L and T_U each form a nested sequence.*

For example, T_{bad} in Figure 6a is the disjoint union of the blue T_L and red T_U.

Definition 3.9 (Tubing decomposition)**.** Let $T_L = \{\tau_1, \tau_2, \ldots\}$ where $\tau_i \subset \tau_{i+1}$ for all i. For convenience, we define $\tau_0 = \emptyset$. We define a nested sequence $\mathcal{L} = (L_1, L_2, \ldots)$ and a sequence of disjoint sets $\mathcal{M}_L = (M_L^1, M_L^2, \ldots)$ as follows.

- For each $i \geq 1$, let $L_i = \tau_i - S$, and mark L_i with a star if $(\tau_i - \tau_{i-1}) \cap S \neq \emptyset$.
- If L_i is the j-th starred set, let $M_L^j = (\tau_i - \tau_{i-1}) \cap S$.

We define the sequences $\mathcal{U}$ and $\mathcal{M}_U$ analogously. We make the following definitions.

- Let $\hat{M} := S - \bigcup_{\tau \in T_{\text{bad}}} \tau$.
- For sequences $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_q)$, let the sequence $\mathbf{a} \cdot \mathbf{b} := (a_1, \ldots, a_p, b_1, \ldots, b_q)$ be their concatenation, and let $\overline{\mathbf{a}} := (a_p, \ldots, a_1)$ be the reverse of $\mathbf{a}$.
- We define
$$\mathcal{M} := \begin{cases} \mathcal{M}_L \cdot \overline{\mathcal{M}_U} & \text{if } \hat{M} = \emptyset \\ \mathcal{M}_L \cdot (\hat{M}) \cdot \overline{\mathcal{M}_U} & \text{if } \hat{M} \neq \emptyset \end{cases}$$
where $(\hat{M})$ is the sequence containing exactly one set: $\hat{M}$.
- The *decomposition* of T_{bad} is the triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$.

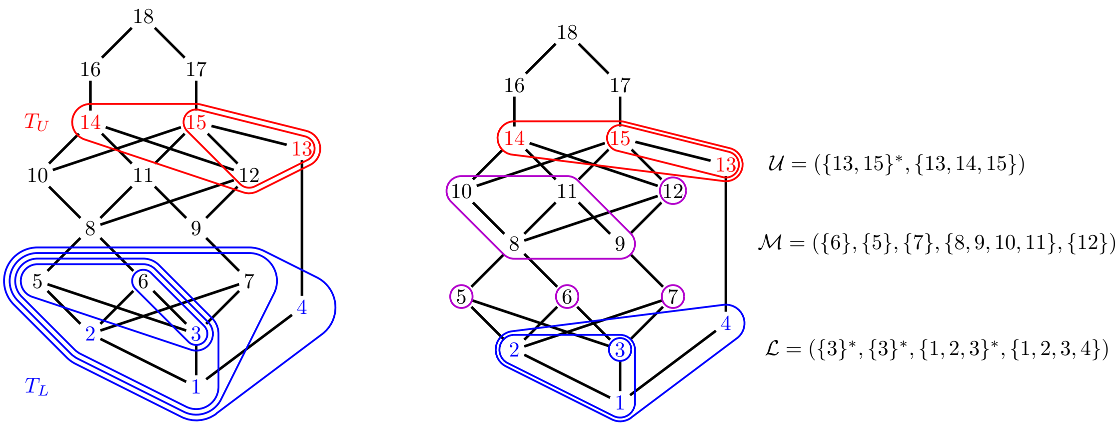
Example 3.10. Figure 6 gives an example of a decomposition.

Lemma 3.11 (Reconstruction algorithm)**.** *T_{bad} can be reconstructed from its decomposition.*

Proof. Let $\mathcal{M} = (M_1, \ldots, M_n)$. To reconstruct T_L, we set $\tau_1 = L_1 \cup M_1$ and take
$$\tau_i = \begin{cases} \tau_{i-1} \cup L_i & \text{if } L_i \text{ is not starred} \\ \tau_{i-1} \cup L_i \cup M_j & \text{if } L_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

For T_U, we set $\tau_1 = U_1 \cup M_n$ and
$$\tau_i = \begin{cases} \tau_{i-1} \cup U_i & \text{if } U_i \text{ is not starred} \\ \tau_{i-1} \cup U_i \cup M_{n-j+1} & \text{if } U_i \text{ is marked with the } j\text{-th star.} \end{cases}$$

$\square$



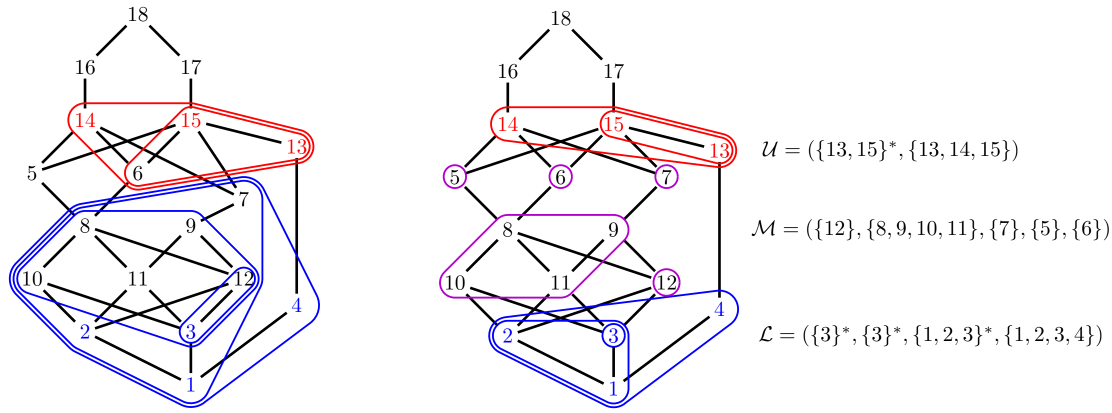
(a) T_L is blue and T_U is red.

(b) $\mathcal{L}$ is blue, $\mathcal{M}$ is purple, and $\mathcal{U}$ is red.

Figure 6: The decomposition of T_{bad}.

Lemma 3.12. *Applying the reconstruction algorithm to* $(\mathcal{L}, \overline{\mathcal{M}}, \mathcal{U})$ *yields a proper tubing* T'_{bad} *of* P' *with exactly* $|T_{bad}|$ *tubes.*

Example 3.13. Figure 7 shows the tubes in T'_{bad} and its $\mathcal{L}, \mathcal{M}, \mathcal{U}$. One can check that the set $\mathcal{M}$ for T'_{bad} (in Figure 7b) is the reverse of the set $\mathcal{M}$ for T_{bad} (in Figure 6b). Furthermore, the number of tubes in T'_{bad} (in Figure 7a) is the same as that in T_{bad} (in Figure 6a).



(a) T'_{bad}

(b) $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{U}$ of T'_{bad}

Figure 7: The decomposition of T'_{bad}.

Finally, we define $T' := T'_{\text{bad}} \sqcup T_{\text{good}}$ and take $\Phi_{P,S}(T) := T'$.

Lemma 3.14. T' *is a proper tubing of* P'*. Furthermore,* $\Phi_{P',S}(T') = T$ *and* $|\Phi_{P,S}(T)| = |T|$.

This completes the proof of Theorem 3.1.

4 h-vector of broom posets and stack-sorting

4.1 Broom posets

The *ordinal sum* of two posets $(P, <_P)$ and $(Q, <_Q)$ is the poset $(R, <_R)$ whose elements are those in $P \cup Q$, and $a \leq_R b$ if and only if

- $a, b \in P$ and $a \leq_P b$ or
- $a, b \in Q$ and $a \leq_Q b$ or
- $a \in P$ and $b \in Q$.

We denote the ordinal sum of P and Q as $P \oplus Q$. Let C_n be the chain poset of size n and A_k be the antichain of size k. In this section, we study the *broom posets* $A_{n,k} = C_{n+1} \oplus A_k$. In particular, $A_{n,0}$ is the chain poset C_{n+1}, and $A_{0,k}$ is the claw poset $C_1 \oplus A_k$. Recall that $\mathscr{A}(A_{n,0})$ is the associahedron and $\mathscr{A}(A_{0,k})$ is the permutohedron (see Figure 4). Hence, the poset associahedra of broom posets interpolate between the classical permutohedra and associahedra. Hence, one may expect a general combinatorial interpretation of the face numbers of these poset associahedra that generalizes that of both permutahedra and associahedra. Indeed, it was shown in [NS23a] that the h-vector of the poset associahedra of broom posets counts descents of stack-sorting preimages.

Stack-sorting is a function $s : \mathfrak{S}_n \to \mathfrak{S}_n$ which attempts to sort the permutations w in $\mathfrak{S}_n$ in linear time, not always sorting them completely (see definition in Section 4.3). A permutation $w \in \mathfrak{S}_n$ is stack-sortable if $s(w) = 12\ldots n$. It is well-known that the h-vector of the classical associahedra counts descents of stack-sortable permutations. In the more general case of poset associahedra of broom posets, we have the following result.

Theorem 4.1 ([NS23a, Theorem 4.8])**.** *Let $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$ and $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then h_i counts the number of permutations in $s^{-1}(\mathfrak{S}_{n,k})$ with exactly i descents.*

In the next few sections, we will summarize the proof of Theorem 4.1 in [NS23a]. The main idea is to use a "third party" set $\mathfrak{P}_{n,k}$ (defined in Section 4.2). Then, in Section 4.3.2 and 4.4.4, we will describe descent-preserving bijections from $s^{-1}(\mathfrak{S}_{n,k})$ and $\mathcal{B}$-trees, an object counted by the h-vector of $\mathscr{A}(A_{n,k})$, to $\mathfrak{P}_{n,k}$, thus proving Theorem 4.1.

We also want to point out the following result by Brändén.

Theorem 4.2 ([Brä08])**.** *For $A \subseteq \mathfrak{S}_n$, we have*

$$\sum_{\sigma \in s^{-1}(A)} x^{\mathrm{des}(\sigma)} = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{|\{\sigma \in s^{-1}(A) \ : \ peak(\sigma) = m\}|}{2^{n-1-2m}} x^m (1+x)^{n-1-2m},$$

where $peak(\sigma)$ is the number of index i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.

Hence, Theorem 4.1 gives the following corollary.

Corollary 4.3. *The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.*

4.2 Catalan convolution

The *Catalan numbers*, $C_n = \frac{1}{n+1}\binom{2n}{n}$, are one of the most well-known sequences in combinatorics. Among hundreds of objects counted by the Catalan numbers, three well-known objects are binary trees, *stack-sortable permutations*, and *Dyck paths*.

A Dyck path of length $2n$ is a path from $(0, 0)$ to (n, n) with steps $(1, 0)$ (up steps) and $(0, 1)$ (right steps) that never goes below the diagonal line. There is a bijection between Dyck paths of length $2n$ and binary trees with n nodes as follows. For a binary tree T:

1. Create a binary tree T' by adding one child to every node in T that has exactly one child, and adding two children to every node in T that has no child. T' is a *full binary tree*, i.e. a binary tree in which each node has zero or two children, and T' has $2n+1$ nodes. The added nodes are the *leaves* of T', and the original nodes in T are the *internal nodes* of T'.

2. Read T' in *preorder*: first read the root, then read the left subtree in preorder before reading the right subtree in preorder. When we read an internal node, add an up step to the Dyck path. When we read a leaf, add a right step. Note that we always ignore the final leaf since there are $2n+1$ nodes in T' but only $2n$ steps in the Dyck path.

Recall that a valley in a Dyck path is a rightstep followed by an upstep. Observe that in the above bijection, the number of right edges in T is the same as the number of valleys in the corresponding Dyck path. For example, in Figure 8, the binary tree has 2 right edges and the corresponding Dyck path has 2 valleys.

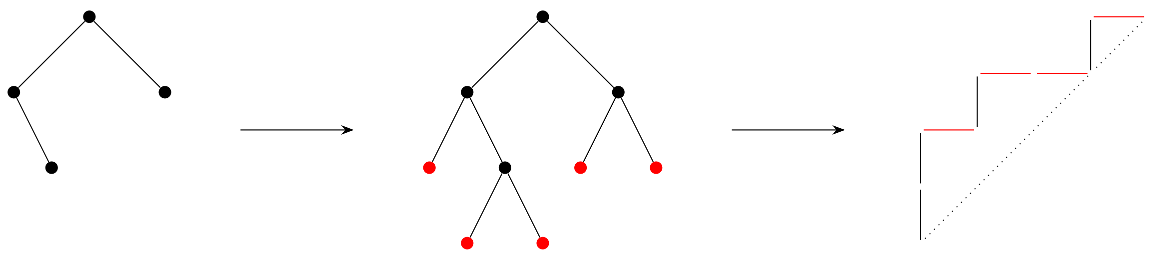


Figure 8: Example of the bijection between binary trees and Dyck paths

The Catalan convolution is defined as follows.

Definition 4.4. For $n, k \in \mathbb{Z}_{\geq 0}$, the kth Catalan convolution is

$$C_n^{(k)} = \sum_{\substack{a_1+a_2+\ldots+a_{k+1}=n \\ a_1,a_2,\ldots,a_{k+1}\in\mathbb{Z}_{\geq 0}}} C_{a_1}C_{a_2}\ldots C_{a_{k+1}}.$$

The explicit formula for $C_n^{(k)}$ is

$$C_n^{(k)} = \frac{k+1}{n+k+1}\binom{2n+k}{n}.$$

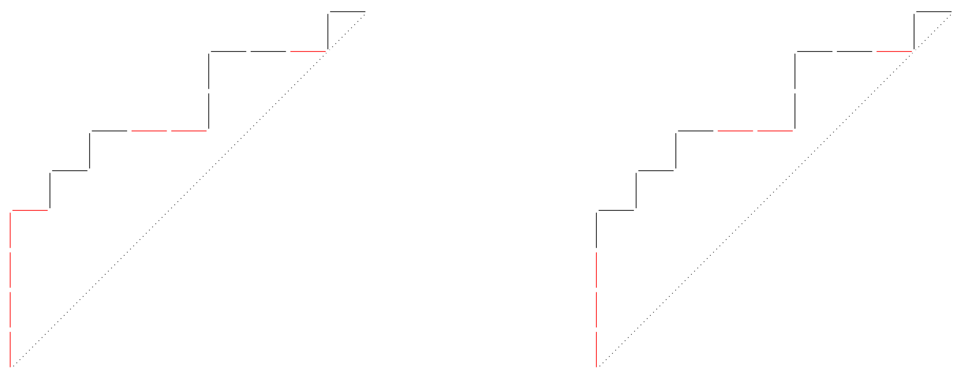
See [Reg12] for a proof of the formula, and [Ted11] for some combinatorial interpretations. By definition, $C_n^{(0)} = C_n$ and $C_n^{(1)} = C_{n+1}$. Also, for all k, we have $C_0^{(k)} = 1$ and $C_1^{(k)} = k+1$. We will use the following combinatorial interpretation of Catalan convolution: $C_n^{(k)}$ counts the number of Dyck paths of length $2(n+k)$ that start with at least k up steps. To see that this is the correct interpretation, recall that a Dyck path of length $2(n+k)$ starting with at least k up steps corresponds to a parenthesization of $n+k$ pairs of parentheses starting with at least k open brackets. We mark these open brackets. For each marked open bracket, we mark the close bracket that matches it. This gives k marked close brackets. In the Dyck path, we mark the steps corresponding to the marked brackets. Thus, the Dyck path has the following form:

$$\underbrace{U,\ldots,U,U}_{k\text{ up steps}}, D_1, R, D_2, R, \ldots, D_k, R, D_{k+1},$$

where the marked steps are colored blue. Observe that the steps in D_1 correspond to the brackets inside the inner-most pair of marked brackets. These brackets have to form a parenthesization. Thus, D_1 is a Dyck path of length $2a_1 \geq 0$. Similarly, each D_i is a Dyck path of length $2a_i \geq 0$. Note that some D_i may have length zero, so we may have consecutive marked right steps.

Definition 4.5. For $n, k \geq 0$, we define $\mathfrak{D}_{n,k}$ to be the set of all Dyck paths of length $2(n+k)$ that start with k up steps. For each Dyck path $D \in \mathfrak{D}_{n,k}$, let $c(D)$ be the vector where c_i is the length of the ith block of consecutive marked right steps.

Thus, $c(D)$ is a composition of k and also depends on k. For example, Figures 9a and 9b both show the same Dyck path D. However, in Figure 9a, we view D as an element of $\mathfrak{D}_{5,4}$, so $c(D) = (1, 2, 1)$, which is a composition of 4. In Figure 9b, we view D as an element of $\mathfrak{D}_{6,3}$, so $c(D) = (2, 1)$, which is a composition of 3.



(a) A Dyck path in $\mathfrak{D}_{5,4}$ (b) A Dyck path in $\mathfrak{D}_{6,3}$

Figure 9: The same Dyck path but viewed as an element of two different sets

Given a permutation $w \in \mathfrak{S}_k$ and a composition $c = (c_1, \ldots, c_\ell)$ of k, c divides the indices $1, 2, \ldots, k$ into ℓ blocks: the first block consists of the indices $1, 2, \ldots, c_1$; the second block consists of the indices $c_1 + 1, c_1 + 2, \ldots, c_1 + c_2$; and so on. We define the *descent of w with respect to c* as

$$\mathrm{des}_c(w) = |\{i \mid i \text{ and } i+1 \text{ are in the same block divided by } c \text{ and } w_i > w_{i+1}\}|.$$

For example, $\mathrm{des}_{(2,2)}(4312) = 1$ because even though $w_2 > w_3$, 2 and 3 are not in the same block divided by $(2, 2)$, so this descent does not count.

Definition 4.6. For $n, k \geq 0$, we define

$$\mathfrak{P}_{n,k} = \{(w, D) \mid w \in \mathfrak{S}_k, D \in \mathfrak{D}_{n,k}\}.$$

For each pair $(w, D) \in \mathfrak{P}_{n,k}$, we define

$$\mathrm{des}(w, D) = \mathrm{des}_{c(D)}(w) + \#\text{valley in } D.$$

4.3 Stack-sorting

4.3.1 Definition

First introduced by Knuth in [K$^+$73], the *stack-sorting algorithm* led to the study of *pattern avoidance* in permutations. In [Wes90], West defined a deterministic version of Knuth's stack-sorting algorithm, which we call the *stack-sorting map* and denote by s. The stack-sorting map is defined as follows.

Definition 4.7 (Stack-sorting)**.** Given a permutation $\pi \in \mathfrak{S}_n$, $s(\pi)$ is obtained through the following procedure. Iterate through the entries of π. In each iteration,

- if the stack is empty or the next entry is smaller than the entry at the top of the stack, push the next entry to the top of the stack;
- else, pop the entry at the top of the stack to the end of the output permutation.

Example 4.8. Figure 10 illustrates the stack-sorting process on $\pi = 3142$.

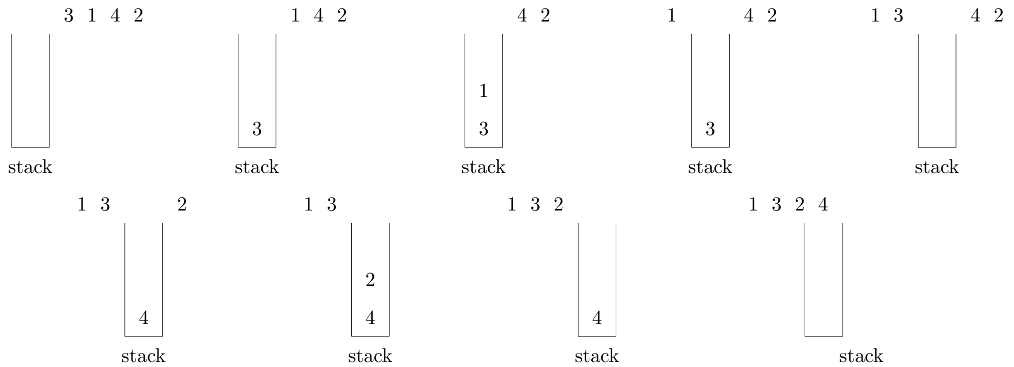


Figure 10: Example of $s(3142)$

Another way to define s is by *decreasing binary trees*. Recall that a *binary tree* is a rooted tree in which each node has at most 2 children, usually called the left and right child. A decreasing binary tree is a binary tree whose n nodes have been labeled bijectively with the numbers $\{1, 2, \dots, n\}$, such that the number in each node is larger than the numbers in its children.

There is a natural bijection between decreasing binary trees of size n and permutations in $\mathfrak{S}_n$ by *inorder reading*. To read a binary tree in inorder, first we read the left subtree in inorder. Then we read the root, and finally we read the right subtree in inorder. Note that this is a recursive definition. For a decreasing binary tree T, we denote by $\mathcal{I}(T)$ the permutation obtained by reading T in inorder. Recall that a *descent* of a permutation w is an index i such that $w_i > w_{i+1}$. Notice that for every decreasing binary tree T, the descents of $\mathcal{I}(T)$ are in one-to-one correspondence with the right edges of T.

Another order to read a binary tree is *postorder*. To read a binary tree in postorder, first we read the left subtree in postorder. Then we read the right subtree in order before we read the root. This is also a recursive definition. For a decreasing binary tree T, we denote by $\mathcal{P}(T)$ the permutation obtain by reading T in postorder.

Example 4.9. Figure 11 shows two permutations obtained by reading a binary tree in inorder and postorder.

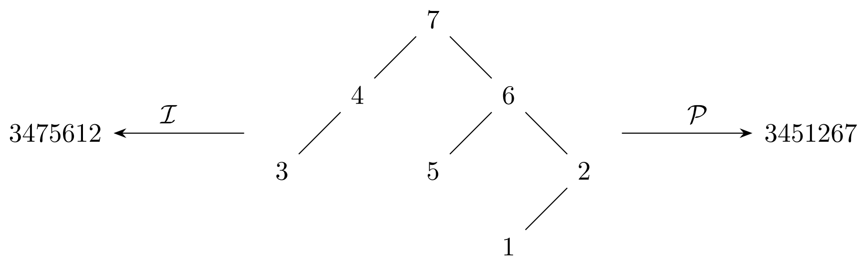


Figure 11: Reading a binary tree in inorder and postorder

The reading orders of binary trees give an alternate definition of stack-sorting.

Proposition 4.10 ([Bón22, Corollary 8.26])**.** *For any $\pi \in \mathfrak{S}_n$, one has*

$$s(\pi) = \mathcal{P}(\mathcal{I}^{-1}(\pi)).$$

For example, we encourage the the readers to check that $s(3475612) = 3451267$, which matches the example in Figure 11.

4.3.2 Descents

We now describe the bijection between $s^{-1}(\mathfrak{S}_{n,k})$ and $\mathfrak{P}_{n,k}$ that preserves the number of descents, as defined in [NS23a, Section 3].

For $w \in s^{-1}(\mathfrak{S}_{n,k})$, let $T = \mathcal{I}^{-1}(w)$ be the decreasing binary tree corresponding to w. Define the *core tree* of T to be the induced subtree of T formed by the nodes $k+1, \ldots, n+k$. Note that the core tree of T is connected, and the nodes have to be labeled from $k+1$ to $n+k$ in postorder. A node in the core tree is *marked* if it contains node $k+1$ in its right subtree. Let $a_1, \ldots, a_{\ell-1}$ be the marked nodes. Observe that since they all contain $k+1$ in their right subtree, they are totally ordered $k+1 = a_\ell <_T a_{\ell_1} <_T \ldots <_T a_1$. Recall that when reading T in postorder, we obtain a permutation in $\mathfrak{S}_{n,k}$. In particular, the nodes appearing before node $k+1$ in postorder are exactly the nodes $1, \ldots, k$. Thus, the nodes in $T_{\leq k+1}$ and in the left subtree of the marked nodes are exactly the nodes $1, \ldots, k$.

Define the sequence $c(T)$ as follows: for $1 \leq i \leq \ell$, c_i equals the number of nodes in the left subtree of a_i in T; in addition, $c_{\ell+1}$ equals the number of nodes in the right subtree of $a_\ell = k+1$ in T. The nodes in the left subtrees of a_i's and in the right subtree of $k+1$ are exactly the nodes $1, \ldots, k$, so $c(T)$ is a weak composition of k.

For each marked node, we now remove its right edge. This divides the core tree of T into ℓ disjoint trees $B_1, \ldots, B_\ell$, with B_i containing a_i. Furthermore, in B_i, a_i is a leaf, and a_i is the left most node, i.e. the unique path from the root to a_i consists of only left edges. We construct a sequence of Dyck paths $D_1, \ldots, D_\ell$ corresponding to T as follows. For each B_i,

1. let B_i' be $B_i \backslash \{a_i\}$;
2. let D_i' be the Dyck path corresponding to B_i' by the bijection in Section 4.2;
3. let D_i be U, D_i', R.

Observe that each D_i is a Dyck path that never returns to the diagonal. Furthermore, the total length of these Dyck paths is exactly $2n$ since there are n nodes in the core tree of T. Now we are ready to state our bijection.

Definition 4.11. Define the map

$$f_{n,k} : s^{-1}(\mathfrak{S}_{n,k}) \to \mathfrak{P}_{n,k}$$

as follows. For $w \in s^{-1}(\mathfrak{S}_{n,k})$, let $T = \mathcal{I}^{-1}(w)$. Let $D_1, \ldots, D_\ell$ be the sequence of Dyck paths corresponding to T, and let $c(T) = (c_1, \ldots, c_{\ell+1})$. We have $f_{n,k}(w) = (\omega, D)$, where

- ω is obtained by removing all numbers $k+1, \ldots, n$ in w, and
- D has the form

$$\underbrace{U, \ldots, U}_{k \text{ up steps}}, \underbrace{R, \ldots, R}_{c_{\ell+1} \text{ right steps}}, D_\ell, \underbrace{R, \ldots, R}_{c_\ell \text{ right steps}}, D_{\ell-1}, \ldots, \underbrace{R, \ldots, R}_{c_2 \text{ right steps}}, D_1, \underbrace{R, \ldots, R}_{c_1 \text{ right steps}}.$$

Note that another way to get ω is to read the nodes $1, \ldots, k$ in T in inorder. Furthermore, $c(T)$ is a weak composition of k, and the total length of $D_1, \ldots, D_\ell$ is $2n$. Thus, D is a Dyck path of length $2(n+k)$ starting with k up steps, i.e. $D \in \mathfrak{D}_{n,k}$. Therefore, $f_{n,k}(w)$ is indeed in $\mathfrak{P}_{n,k}$ since $\omega \in \mathfrak{S}_k$, and $D \in \mathfrak{D}_{n,k}$.

Example 4.12. Let us show an example of this map. Figure 12 shows a binary tree T with $\mathcal{I}(T) \in s^{-1}(\mathfrak{S}_{11,6})$. The marked nodes of T are colored red, i.e. $a_1 = 13$, $a_2 = 11$, $a_3 = 10$, and $a_4 = 7$. Thus, we have $c_1 = 2$ since there are two nodes in the left subtree of $a_1 = 13$. Similarly, $c_2 = 0$, $c_3 = 1$, $c_4 = 2$. Finally, $c_5 = 1$ since there is one node in the right subtree of $a_4 = 7$.

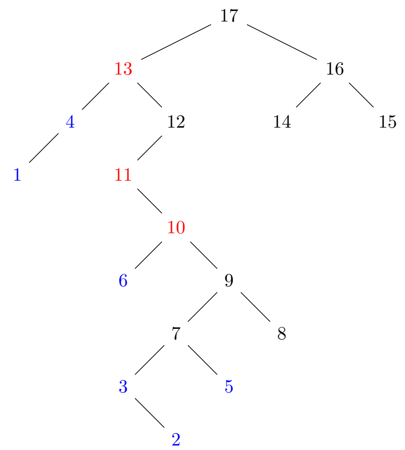


Figure 12: A binary tree T with $\mathcal{I}(T) \in s^{-1}(\mathfrak{S}_{11,6})$

Next, removing the right edges of a_i for $1 \le i < 4$, we obtain four disjoint binary trees shown in Figure 13. Figure 13 also shows the corresponding Dyck paths. Observe that these are Dyck paths that never return to the diagonal (until the last step).

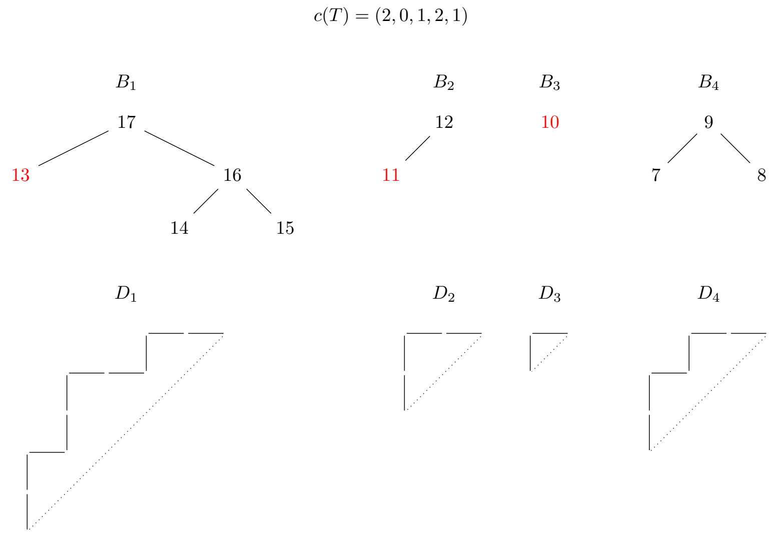


Figure 13: The sequence $c(T)$, the disjoint binary trees and the corresponding Dyck paths

Putting the Dyck paths and $c(T)$ together, we obtain the Dyck path in Figure 14.

Proposition 4.13. *The map $f_{n,k}$ above is a bijection.*

In particular, we can easily find the size of $s^{-1}(\mathfrak{S}_{n,k})$.

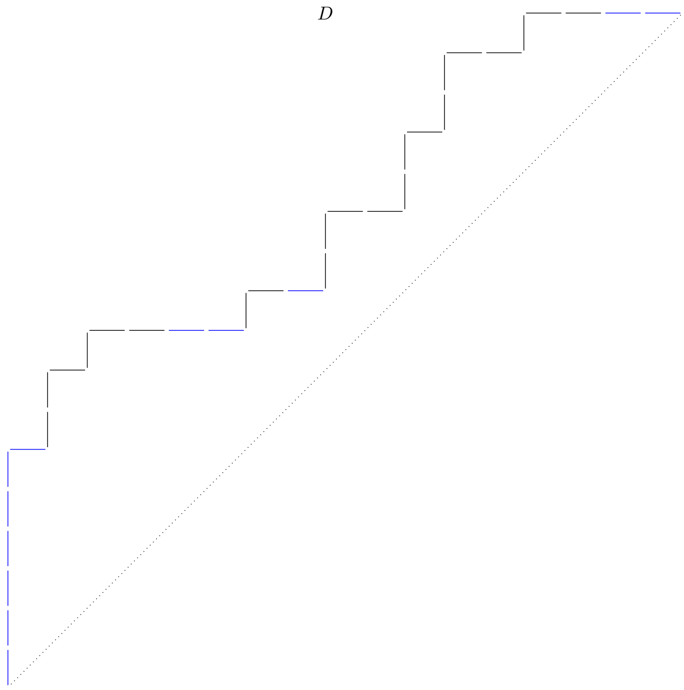


Figure 14: The pair $(\omega, D) = f_{11,6}(T)$

Corollary 4.14. *For all $n, k \geq 0$, we have*

$$|s^{-1}(\mathfrak{S}_{n,k})| = k! \cdot C_n^{(k)}.$$

Recall that $C_k^{(0)} = C_n$. Thus, setting $k = 0$ in Corollary 4.14, we recover the well-known result that $|s^{-1}(12 \dots n)| = C_n$.

Proposition 4.15. *For any $w \in s^{-1}(\mathfrak{S}_{n,k})$, we have*

$$\mathrm{des}(w) = \mathrm{des}(f(w)).$$

4.4 Graph associahedra and $\mathcal{B}$-trees

4.4.1 Graph associahedra

It turns out that the poset associahedra of broom posets are also graph associahedra. It is actually more convenient to study them as graph associahedra since there is a known combinatorial interpretation of the h-vector of graph associahedra. Graph associahedra are generalized permutohedra arising as special cases of nestohedra. We refer the readers to [PRW06] for a comprehensive study of face numbers of generalized permutohedra and nestohedra.

Definition 4.16. Let $G = (V, E)$ be a connected graph, and $\tau, \sigma \subseteq V$ be subsets of vertices.

- τ is a *tube* of G if $\tau \neq V$ and it induces a connected subgraph of G.
- τ and σ are *nested* if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$. τ and σ are *disjoint* if $\tau \cap \sigma = \emptyset$.
- τ and σ are *compatible* if they are nested or they are disjoint and $\tau \cup \sigma$ is not a tube.

- A *tubing* T of G is a set of pairwise compatible tubes.
- A tubing T is *maximal* if it is maximal by inclusion, i.e. T is not a proper subset of any other tubing.

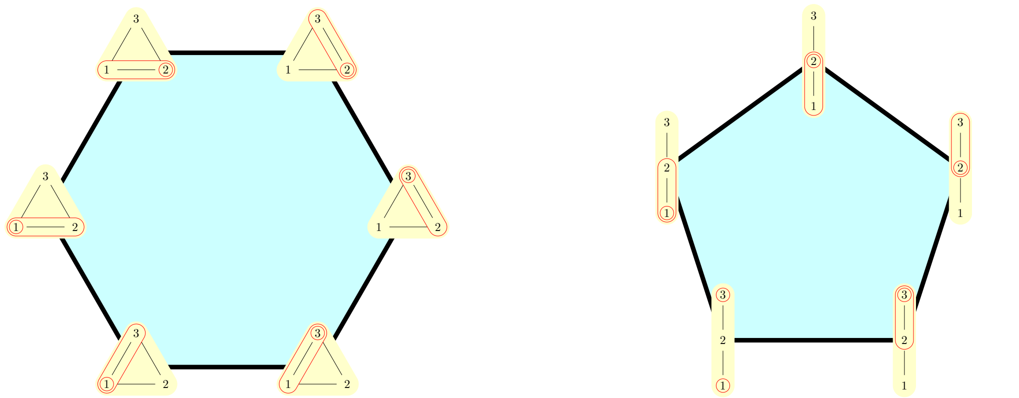
Example 4.17. Figure 15 shows examples and non-examples of tubings of graphs. Note that the left-most example in Figure 15b is a non-example since the tubes $\{1\}$ and $\{4\}$ are disjoint yet their union $\{1, 4\}$ is still a tube. The same reason applies for the right-most example.

(a) Examples (b) Non-examples

Figure 15: Examples and non-examples of tubings of graphs

Definition 4.18. For a connected graph $G = (V, E)$, the **graph associahedron** of G is a simple, convex polytope $\mathsf{Ass}(G)$ of dimension $|V| - 1$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. The faces of $\mathsf{Ass}(G)$ correspond to tubings of G, and the vertices of $\mathsf{Ass}(G)$ correspond to maximal tubings of G.

Example 4.19. Examples of graph associahedra can be seen in Figure 16. In particular, if G is a complete graph, $\mathsf{Ass}(G)$ is a permutohedron. If G is a path graph, $\mathsf{Ass}(G)$ is an associahedron.



(a) Permutohedron (b) Associahedron

Figure 16: Permutohedron and associahedron as graph associahedra

4.4.2 Graph associahedra and poset associahedra

Despite the similarity between graph associahedra and poset associahedra, neither of them is a subset of the other. Nevertheless, when the Hasse diagram of a poset P is a tree, let G_P be the line graph of the Hasse diagram of P, then $\mathscr{A}(P)$ is isomorphic to $\mathsf{Ass}(G_P)$. For instance, if P is a claw, then G_P is a complete graph, and $\mathscr{A}(P)$ and $\mathsf{Ass}(G_P)$ are both permutohedra. If P is a chain, then G_P is a path graph, and $\mathscr{A}(P)$ and $\mathsf{Ass}(G_P)$ are both associahedra. One can see a clear correspondence betweem tubings of P and G_P in Figures 4 and 16.

Conveniently, the Hasse diagrams of broom posets are trees, so their poset associahedra are also graph associahedra. An (n, k)-lollipop graph, denoted $L_{n,k}$, is a graph consisting of a path graph of size n and a complete graph of size k, connected by an edge. We call the unique vertex in the complete graph that is adjacent to the path graph the *link vertex*. We call the other vertices in the complete graph the *clique vertices*. We call the other vertices in the path graph the *path vertices*. For instance, in Figure 17, the link vertex is colored blue, and the clique vertices are colored red.

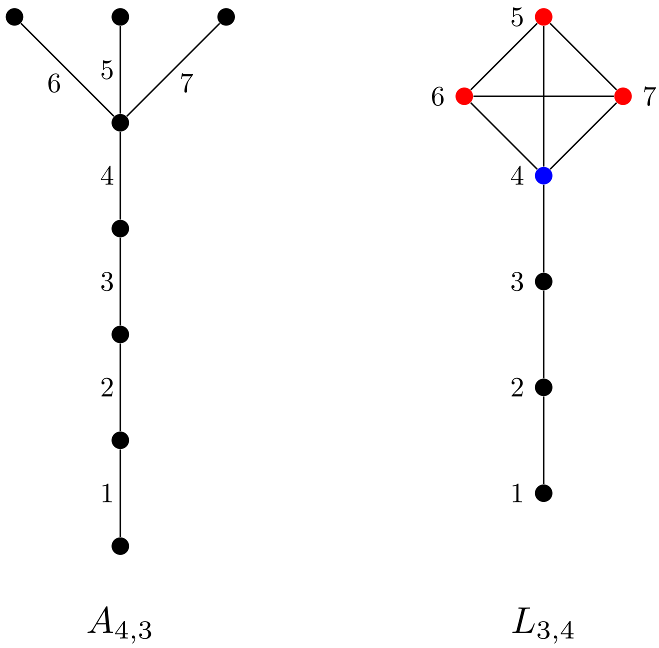


Figure 17: Poset $A_{4,3}$ and graph $L_{3,4}$

Observe that the line graph of the Hasse diagram of $A_{n,k}$ is $L_{n-1,k+1}$. For example, Figure 17 shows the correspondence between the edges of the Hasse diagram of $A_{4,3}$ and the vertices of $L_{3,4}$. This means $\mathscr{A}(A_{n,k})$ is isomorphic to $\mathsf{Ass}(L_{n-1,k+1})$. Therefore, instead of studying the h-vector of $\mathscr{A}(A_{n,k})$, we will study the h-vector of $\mathsf{Ass}(L_{n-1,k+1})$.

4.4.3 $\mathcal{B}$-trees

Every maximal tubing of G can be associated with a $\mathcal{B}$*-tree*. Recall that a *rooted tree* is a tree with a distinguished node, called its *root*. One can view a rooted tree T as a partial order on its nodes in which $i <_T j$ if j lies on the unique path from i to the root. For a node i in a rooted tree T, let $T_{\leq i} = \{j \mid j \leq_T i\}$ be the set of all descendants of i. Note that $i \in T_{\leq i}$. Nodes i and j in a rooted tree are called *incomparable* if neither i is a descendant of j, nor j is a descendant of i. A *descent* of T is an edge $(i, j) \in E$ such that $i < j$ and $j <_T i$. We denote $\operatorname{des}(T)$ the number of descents in T.

Definition 4.20. For a maximal tubing $\mathcal{B}$ of a graph $G = ([n], E)$, its $\mathcal{B}$*-tree* is a rooted tree T on the node set $[n]$ such that

- For any $i \in [n]$ such that i is not the root, one has $T_{\leq i} \in \mathcal{B}$.
- For $k \geq 2$ incomparable nodes $i_1, \ldots, i_k \in [n]$, one has $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$.

Example 4.21. Figure 18 shows three $\mathcal{B}$-trees corresponding to three maximal tubings of a path graph. It is clear that $\mathcal{B}$-trees of the same graph are not necessarily isomorphic.

The h-polynomial of $\mathsf{Ass}(G)$ is counted by the descents of the $\mathcal{B}$-trees.

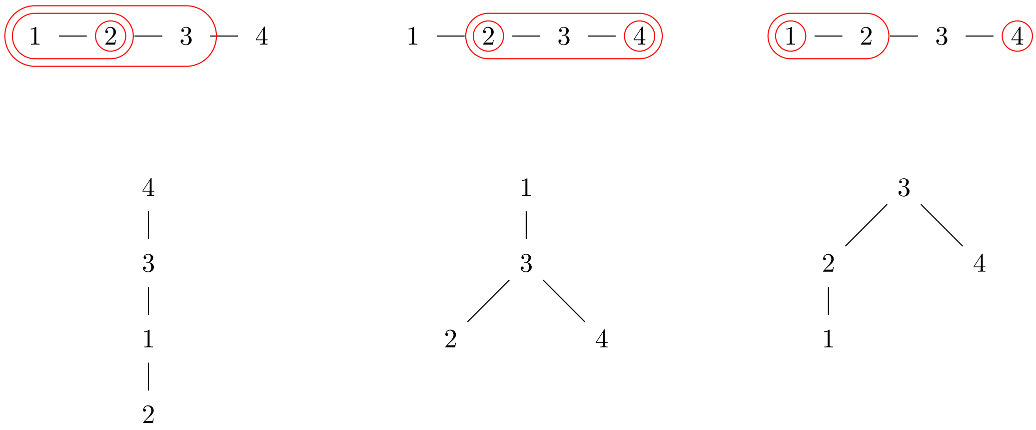


Figure 18: Maximal tubings of a path graph and their corresponding $\mathcal{B}$-trees

Theorem 4.22 ([PRW06, Corollary 8.4]). *For a connected graph G, the h-polynomial of $\mathsf{Ass}(G)$ is given by*

$$h_{\mathsf{Ass}(G)}(t) = \sum_{T} t^{\operatorname{des}(T)},$$

where the sum is over all $\mathcal{B}$-trees T.

4.4.4 Descents

Recall from Section 4.4.2 that the poset associahedra of $A_{n,k}$ is also the graph associahedra of the lollipop graph $L_{n-1,k+1}$. Let $\mathfrak{B}_{n-1,k+1}$ be the set of $\mathcal{B}$-trees of $L_{n-1,k+1}$. We now described the descent-preserving bijection between $\mathfrak{B}_{n-1,k+1}$ and $\mathfrak{P}_{n,k}$, as defined in [NS23a, Section 4].

First, we will label the vertices in $L_{n-1,k+1}$ as follows. We label the link vertex n. We label the clique vertices $n+1, \dots, n+k$. Finally, we label the path vertices $n-1, \dots, 1$ in decreasing order starting from vertex n. Figure 19 shows an example of this labeling for $L_{11,4}$.

First, let us make some observations about the $\mathcal{B}$-trees in $\mathfrak{B}_{n-1,k+1}$. Our running example throughout these observations will be Figure 19.

Lemma 4.23. *Let $B \in \mathfrak{B}_{n-1,k+1}$. The nodes $n+1, \dots, n+k$ are totally ordered.*

Lemma 4.23 means that we have a chain $w_1 <_T w_2 <_T \dots <_T w_k$ where $w_i \in \{n+1, \dots, n+k\}$. We call the unique path from w_1 to the root the *core chain* of B. Let $a_1 <_T a_2 <_T \dots <_T a_\ell$ be the other nodes in the core chain. For example, in Figure 19, the clique nodes (colored red) are totally ordered $14 <_T 15 <_T 13$. The other elements of the core chain are colored blue.

Lemma 4.24. *The nodes $n+1 \dots, n+k$ have at most one child.*

Lemma 4.24 means that in the core chain of B, the only nodes that may have two children are $a_1, \dots, a_\ell$. For these nodes, we call the branch that contains w_1 their *main branch*. We call the other branch, if exists, their *secondary branch*. For instance, in Figure 19, the clique nodes all have one child. The other nodes in the core chain may or may not have two children. For node 10, which has two children, the secondary branch consists of the nodes $7, 8, 9$.

Lemma 4.25. *We have $a_1 > a_2 > \dots > a_\ell$. Moreover, the secondary branch of a_i contains exactly the nodes $a_{i+1}+1, a_{i+1}+2, \dots, a_i - 1$.*

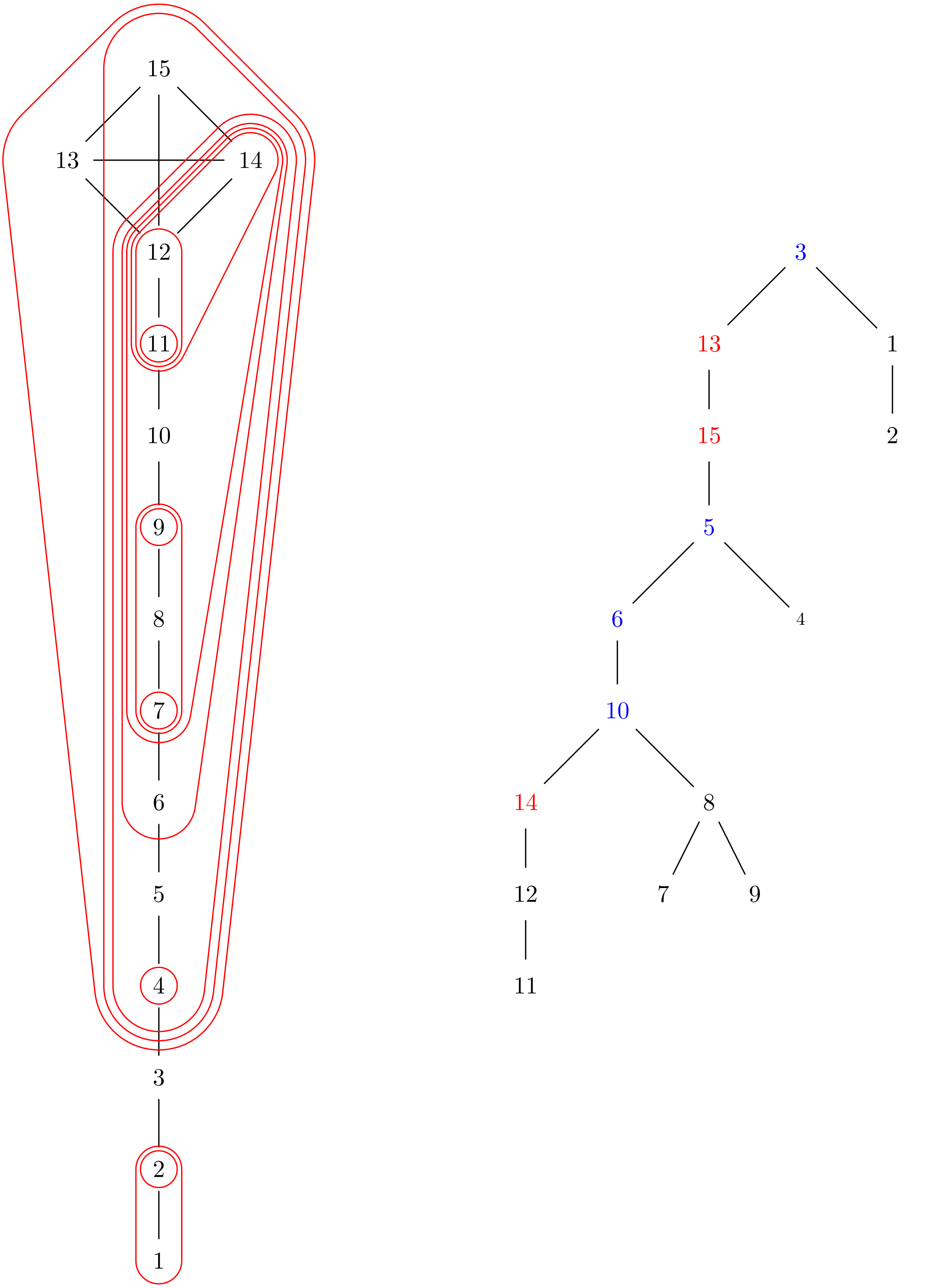


Figure 19: A tubing of $L_{11,4}$ and the corresponding $\mathcal{B}$-tree

For example, in Figure 19, the secondary branch of 10 consists of nodes $7, 8, 9$, which are exactly the numbers between $a_1 = 10$ and $a_2 = 6$.

Lemma 4.26. *If w_1 has a child, then $T_{<w_1} = \{n, n-1, \ldots, a_1 + 1\}$.*

Back to our running example, in Figure 19, the descendants of $w_1 = 14$ are 11 and 12, which are exactly the numbers from $n = 12$ to $a_1 + 1 = 11$.

Lemmas 4.25 and 4.26 means that the descendants of w_1 form a $\mathcal{B}$-tree B_0 of the subgraph $(a_1+1)-(a_1+2)-\ldots-n$. This subgraph is a path graph of $n-a_1$ elements. Similarly, the secondary branch of each a_i forms a $\mathcal{B}$-tree B_i of the subgraph $(a_{i+1}+1)-(a_{i+1}+2)-\ldots-(a_i-1)$. This is also a path graph of $a_i-a_{i+1}-1$ elements (with $a_{\ell+1}=0$).

In [PRW06, Section 10.2], it is shown that there is a bijection between $\mathcal{B}$-trees of path graphs and binary trees. Moreover, the descent edges of the $\mathcal{B}$-trees correspond to the right edges of the binary trees. This means that there is a bijection between $\mathcal{B}$-trees of path graphs and Dyck paths such that the descent edges of the $\mathcal{B}$-trees correspond to the valleys of the Dyck paths.

Let B_0 be the tree formed by the descendants of w_1. For $1 \leq i \leq \ell$, let B_i be the tree formed by the secondary branch of a_i. Next, we construct a sequence of Dyck paths $D_1, \ldots, D_\ell$ as follows. For each B_i with $1 \leq i \leq \ell$,

- let D'_i be the Dyck path corresponding to B_i by the bijection above;
- let D_i be U, D'_i, R.

Once again, each D_i is a Dyck path that never returns to the diagonal. Finally, let D_0 be the Dyck path corresponding to B_0. D_0 is a Dyck path that may return to the diagonal. Furthermore, for $1 \leq i \leq \ell$, D'_i is a Dyck path of length $(a_i - ai + 1 - 1)$, so D_i is a Dyck path of length $2(a_i - a_{i+1})$. D_0 is a Dyck path of length $2(n-a_1)$. Thus, the total length of these Dyck paths is exactly $2n$. Now we are ready to state our bijection.

Definition 4.27. Define the map

$$g_{n,k} : \mathfrak{B}_{n,k} \to \mathfrak{P}_{n,k}$$

as follows. For $B \in \mathfrak{B}_{n,k}$, we construct $w_1, \ldots, w_k$ and $a_1, \ldots, a_\ell$ as above. Let $D_0, D_1, \ldots, D_\ell$ be the sequence of Dyck paths constructed as above. Also, for $1 < i \leq \ell$, let c_i be the number of clique nodes between a_i and a_{i-1}. Let c_1 be the number of clique nodes below a_1 and $c_{\ell+1}$ be the number of clique nodes above a_ℓ. We have $g_{n,k}(B) = (w, D)$, where

- $w = (w_1 - n), (w_2 - n), \ldots, (w_k - n)$, and
- D has the form

$$\underbrace{U, \ldots, U}_{k \text{ up steps}}, D_0, \underbrace{R, \ldots, R}_{c_1 \text{ right steps}}, D_1, \underbrace{R, \ldots, R}_{c_2 \text{ right steps}}, D_2, \ldots, D_{\ell-1}, \underbrace{R, \ldots, R}_{c_\ell \text{ right steps}}, D_\ell, \underbrace{R, \ldots, R}_{c_{\ell+1} \text{ right steps}} .$$

By definition, $c_1 + \ldots + c_{\ell+1}$ is the total number of clique nodes, which is k. The total length of $D_1, \ldots, D_\ell$ is $2n$. Thus, D is a Dyck path of length $2(n+k)$ starting with k up steps, i.e. $D \in \mathfrak{D}_{n,k}$. Clearly, $w \in \mathfrak{S}_k$. Therefore, $g_{n,k}(B)$ is indeed in $\mathfrak{P}_{n,k}$.

Proposition 4.28. *The map $g_{n,k}$ above is a bijection.*

Proposition 4.29. *For any $B \in \mathfrak{B}_{n,k}$, we have*

$$\mathrm{des}(B) = \mathrm{des}(g(B)).$$

4.5 Real-rootedness

In this section, we will use a "happy coincidence" in stack-sorting to show real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$. Recall that we say a polynomial $a_0 + a_1x + \ldots + a_nx^n$ is real-rooted if all of its zeros are real. We say a sequence $(a_0, a_1, \ldots, a_n)$ is real-rooted if its generating function $a_0 + a_1x + \ldots + a_nx^n$ is real-rooted.

Let f and g be real-rooted polynomials with positive leading coefficients and real roots $\{f_i\}$ and $\{g_i\}$, respectively. We say that f *interlaces* g if

$$g_1 \leq f_1 \leq g_2 \leq f_2 \leq \ldots \leq f_{d-1} \leq g_d$$

where $d = \deg g = \deg f + 1$. We say that f *alternates left of* g if

$$f_1 \leq g_1 \leq f_2 \leq g_2 \leq \ldots \leq f_d \leq g_d$$

where $d = \deg g = \deg f$. Finally, we say f *interleaves* g, denoted $f \ll g$, if f either interlaces or alternates left of g.

A classic example of real-rooted polynomials is a Narayana polynomial. Recall that the Narayana polynomial $N_n(x)$ is defined by

$$N_n(x) = \sum_{i=0}^{n-1} a_i x^i$$

where a_i counts the number of permutations in $s^{-1}(12\ldots n)$ with exactly i descents. In other words, $N_n(x)$ is the h-polynomial of $\mathscr{A}(A_{n,0})$ and $\mathscr{A}(A_{n-1,1})$. We have the following result.

Theorem 4.30 ([Brä06])**.** *For all n, $N_n(x)$ is real-rooted. Furthermore, $N_{n-1}(x) \ll N_n(x)$.*

To prove real-rootedness of the h-polynomial of $\mathscr{A}(A_{n,2})$, we will need the following "happy coincidence".

Proposition 4.31. *The number of permutations in $s^{-1}(2134\ldots n)$ with exactly i descents is the same as the number of permutations w in $s^{-1}(1234\ldots n)$ with exactly i descents such that $w_1, w_n < n$.*

Here we sketch the bijection used in [NS23a] to prove Theorem 4.31. Let

$$\mathcal{T}_1 = \{T \mid \mathcal{I}(T) = w \in s^{-1}(1234\ldots n), w_1, w_n < n\}$$

and

$$\mathcal{T}_2 = \{T \mid \mathcal{I}(T) = w \in s^{-1}(2134\ldots n)\}.$$

In addition, for two nodes v_1, v_2 in a binary tree T, we say $v_1 \to_R v_2$ (resp. $\to_L$) if v_1 is the right (resp. left) child of v_2. Our bijection φ is constructed as follows.

Given $T \in \mathcal{T}_1$, let v be the smallest ancestor of node 1 that has two children. Then, we must have a chain $1 \to_{D_1} 2 \to_{D_2} \ldots \to_{D_{v-1}} v$ where each D_i is either R or L, and each node $2, 3, \ldots, v-1$ has exactly one child. Furthermore, since $\mathcal{I}(T) = 1234\ldots n$ and v has two children, 1 has to be in the left-subtree of v, so $D_{v-1} = L$. Then, $\varphi(T) \in \mathcal{T}_2$ is constructed as follows.

1. Remove all nodes below $v-1$.

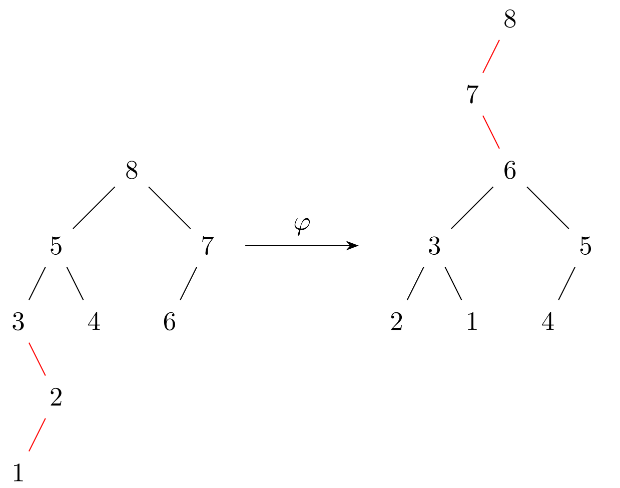


Figure 20: Example of the map φ

2. The root of T has to be n, add the follow edges: $n \to_{D_{v-2}} n+1 \to_{D_{v-3}} \dots \to_{D_1} n+v-2$.
3. Relabel the nodes such that the postorder reading word is $2134\dots n$.

An example of the map φ above can be seen in Figure 20.

Proposition 4.31 gives the following important recurrence.

Proposition 4.32. *Let $H_n(x)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$, and recall that $N_{n+2}(x)$ and $N_{n+1}(x)$ are the h-polynomials of $\mathscr{A}(A_{n+2,0})$ and $\mathscr{A}(A_{n+1,0})$, respectively. We have*

$$H_n(x) = 2N_{n+2}(x) - (1+x)N_{n+1}(x).$$

From the recurrence in Proposition 4.32, and the useful fact that $N_{n+1}(x) \ll N_{n+2}(x)$ in Theorem 4.30, we have the following theorem.

Theorem 4.33. *Let $H_n(x)$ be the h-polynomial of $\mathscr{A}(A_{n,2})$. Then, $H_n(x)$ is real-rooted.*

4.6 Two-leg broom posets

Now we shift our attention to *two-leg broom posets* $A_{2,n,k} = A_2 \oplus C_{n+1} \oplus A_k$. For example, Figure 21 shows the two-leg broom poset $A_{2,3,3}$.

The h-vectors of $\mathscr{A}(A_{2,n,k})$ are also given by stack-sorting preimages.

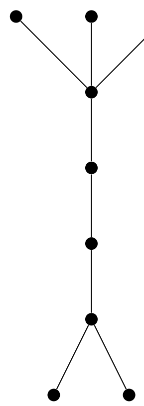
Proposition 4.34. *Let $\mathfrak{S}_{n+3,k} = \{w \mid w \in \mathfrak{S}_{n+k+3}, w_i = i \text{ for all } i > k\}$ and $h = (h_0, h_1, \dots, h_{n+k+1})$ be the h-vector of $\mathscr{A}(A_{2,n,k})$. Then h_i counts the number of permutations in*

$$\{w \mid w \in s^{-1}(\mathfrak{S}_{n+3,k}), w_1 \le n+k+1, w_{n+k+3} \ge n+k+2\}$$

with exactly i descents.

Question 4.35. Are there any other stack-sorting preimages whose descent-generating functions give the h-polynomial of poset associahedra? In particular, one may ask for such interpretation for *many-leg broom posets* $A_{\ell,n,k} = A_\ell \oplus C_{n+1} \oplus A_k$.

In his FPSAC 2023 Extended Abstract, Sack found the following.



$A_{2,3,3}$

Figure 21: A two-leg broom poset

Proposition 4.36. *Let $K_{m,n}$ be the complete bipartite poset $A_m \oplus A_n$. Then*

$$h_i(\mathscr{A}(K_{m,n})) = |\{w \in \mathfrak{S}_{m+n}, \operatorname{des}(w) = i, w_1 \leq m, w_{m+n} \geq m+1\}|,$$

$$\gamma_i(\mathscr{A}(K_{m,n})) = |\{w \in \hat{\mathfrak{S}}_{m+n}, \operatorname{des}(w) = i, w_1 \leq m, w_{m+n} \geq m+1\}|,$$

where $\hat{\mathfrak{S}}_{m+n}$ is the set of permutations in $\mathfrak{S}_{m+n}$ with no double descents or final descent.

Thus, we expect the answer of Question 4.35 for many-leg broom posets to be a generalization of Proposition 4.36. This is indeed the case for Proposition 4.34. Another relevant question is the following.

Question 4.37. Given a (strong) composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, find a combinatorial interpretation for the h-and-γ-vectors for $\mathscr{A}(A_\alpha)$ where $A_\alpha = A_{\alpha_1} \oplus \ldots \oplus A_{\alpha_\ell}$. By Theorem 3.1, it actually suffices to answer this question for partitions λ.

5 An h-vector identity

5.1 Polynomials

We conjecture that the recurrence in Proposition 4.32 can be generalized to any poset. Let us first introduce some relevant polynomials. The (type A) Narayana polynomial is defined to be

$$N_n(x) = \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k} \binom{n}{k+1} x^k.$$

For example, we have

$$\begin{aligned}
N_1(x) &= 1, \\
N_2(x) &= 1 + x, \\
N_3(x) &= 1 + 3x + x^2, \\
N_4(x) &= 1 + 6x + 6x^2 + x^3.
\end{aligned}$$

It is well-known that Narayana polynomials give the h-vectors of the classical associahedra. The corresponding f-vectors are

$$F_n(x) = N_n(x+1).$$

For example, we have

$$\begin{aligned}
F_1(x) &= 1,\\
F_2(x) &= 2 + x,\\
F_3(x) &= 5 + 5x + x^2,\\
F_4(x) &= 14 + 21x + 9x^2 + x^3.
\end{aligned}$$

We also define

$$\tilde{F}_n(x) = nF_{n-1}(x),$$

with the convention that $F_0(x) = 1$. For example, we have

$$\begin{aligned}
\tilde{F}_1(x) &= 1,\\
\tilde{F}_2(x) &= 2,\\
\tilde{F}_3(x) &= 6 + 3x,\\
\tilde{F}_4(x) &= 20 + 20x + 4x^2.
\end{aligned}$$

Similarly, the type B Narayana polynomial is defined to be

$$B_n(x) = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 x^k.$$

For example, we have

$$\begin{aligned}
B_1(x) &= 1,\\
B_2(x) &= 1 + x,\\
B_3(x) &= 1 + 4x + x^2,\\
B_4(x) &= 1 + 9x + 9x^2 + x^3.
\end{aligned}$$

The type B Narayana polynomials show up as the rank-generating function of the type B analogue NC_n^B of the lattice of non-crossing partitions (see [Rei97]) and the h-polynomials of type B associahedra (see [Sim03]). In particular, the sum of the coefficients in $B_{n+1}(x)$ is $\binom{2n}{n}$, which is called type B Catalan number. The corresponding f-vectors of type B associahedra are

$$G_n(x) = B_n(x + 1).$$

For example, we have

$$\begin{aligned}
G_1(x) &= 1,\\
G_2(x) &= 2 + x,\\
G_3(x) &= 6 + 6x + x^2,\\
G_4(x) &= 20 + 30x + 12x^2 + x^3.
\end{aligned}$$

For each family of polynomials $\{P_n(x)\}$, and each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we define

$$P_\lambda(x) = P_{\lambda_1}(x)P_{\lambda_2}(x)\ldots P_{\lambda_\ell}(x).$$

For example, we have

$$\begin{aligned}
N_{(4,2,1)}(x) &= (1 + 6x + 6x^2 + x^3)(1 + x)(1),\\
F_{(4,2,1)}(x) &= (14 + 21x + 9x^2 + x^3)(2 + x)(1),
\end{aligned}$$

$$\tilde{F}_{(4,2,1)}(x) = (20 + 20x + 4x^2)(2)(1),$$
$$B_{(4,2,1)}(x) = (1 + 9x + 9x^2 + x^3)(1 + x)(1),$$
$$G_{(4,2,1)}(x) = (20 + 30x + 12x^2 + x^3)(2 + x)(1).$$

For each permutation w, the cycle type of w is a partition $\lambda(w)$, and the number of cycles in w is $\ell_w = \ell(\lambda(w))$. We abuse notation and define

$$P_w(x) = P_{\lambda(w)}(x)$$

for each family of polynomials $\{P_n(x)\}$. Note that this means P_{w_1} and P_{w_2} are the same if w_1 and w_2 are in the same conjugacy class.

Finally, we denote by $s_{n,k}$ the *unsigned Stirling number of the first kind*, which counts the number of permutations of $\mathfrak{S}_n$ with k cycles. Note that $s_{n,k}$ is the coefficient of x^k in $x(x+1)\dots(x+n-1)$, or equivalently the coefficient of x^{n-k} in $1(1+x)\dots(1+(n-1)x)$.

5.2 Identity

Recall that for a poset P, a subposet S of P is called *autonomous* if there exists a poset Q and $a \in Q$ such that $P = Q(a \to S)$. A subposet S of P is *proper* if $S \neq P$. Our main conjecture is the following.

Conjecture 5.1. *Let P be a poset with a proper autonomous subposet S that is a chain of size n, i.e. $P = Q(a \to C_n)$. For $1 \leq i \leq n$, let P_i be the poset obtained from P by replacing S by an antichain of size i, i.e. $P_i = Q(a \to A_i)$. Let $h_P(x)$, $h_{P_1}(x)$, …, $h_{P_n}(x)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, …, $\mathscr{A}(P_n)$, respectively. Then,*

$$h_P(x) = \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} B_w(x) h_{P_{\ell_w}}(x). \tag{1}$$

In particular, when $n = 2$, we have

$$h_P(x) = \frac{1}{2}\left(h_{P_2}(x) + (1+x)h_{P_1}(x)\right),$$

which give the formula in Proposition 4.32. We will show that Conjecture 5.1 follows from the following conjecture.

Conjecture 5.2. *For all n,*

$$\sum_{w \in \mathfrak{S}_n} t^{\ell_w} G_w(x) = \sum_{w \in \mathfrak{S}_n} t(t+x)\dots(t+(\ell_w - 1)x)\tilde{F}_w(x). \tag{2}$$

Example 5.3. For $n = 3$, the LHS of (2) is

$$t^3 + 3t^2(x+2) + 2t(x^2 + 6x + 6),$$

and the RHS is

$$t(t+x)(t+2x) + 3t(t+x)(2) + 2t(3x+6).$$

One can check that they are equal.

Proposition 5.4. *Conjecture 5.1 follows from Conjecture 5.2.*

We will need a few lemmas to prove Proposition 5.4. Let $P = Q(a \to C_n)$ be a poset with a proper autonomous subposet $S = C_n$. We say a tubing T of P is *degradable* if there is a tube $\tau \in T$ such that $\tau \subseteq S$. We say that T is *non-degradable* otherwise. Our main lemma is the following, which will be proved in Section 5.2.1.

Lemma 5.5. *Let t_k be the number of non-degradable tubings of $P = Q(a \to C_n)$ with k tubes, and $t_{i,k}$, for $1 \le i \le n$, be the number of tubings of $P_i = Q(a \to A_i)$ with k tubes. Then*

$$n!t_k = \sum_{i=1}^{n} s_{n,i} t_{i,k}.$$

On the other hand, if T is degradable, we say a tube τ of T is *degrading* if $\tau \subseteq S$. Clearly, the degrading tubes of T gives a tubing of S. Here we modify the rule slightly and allow S to be a tube of S.

Given a tubing of $S = C_n$, we say a tube is *maximal* if it is not contained in another tube. We say an element $s \in S$ is *lonely* if it is not contained in any tube. Then, the lonely elements and maximal tubes of each tubing gives a composition of n.

Example 5.6. Figure 22 shows a tubing of $S = C_{10}$. The lonely elements and maximal tubes are colored red. The composition is $(2, 1, 1, 3, 3)$.

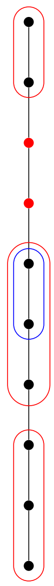


Figure 22: A tubing of $S = C_{10}$

The following lemma is immediate.

Lemma 5.7. *Let T' be a tubing of $S = C_n$, let $(\alpha_1, \dots, \alpha_\ell)$ be the composition corresponding to T'. Then the number of tubings with k tubes of $Q(a \to S)$ that contain T' is the same as the number of non-degradable tubings of $Q(a \to C_\ell)$ with $k - |T'|$ tubes.*

To see Lemma 5.7, from a tubing with k tubes of $Q(a \to S)$ that contain T', one can contract every maximal tube of T' into a single element and obtain a non-degradable tubing of $Q(a \to C_\ell)$ with $k - |T'|$ tubes. Figure 23 gives an example of this contraction.

Combining Lemma 5.5 and 5.7, we have the following lemma.

Lemma 5.8. *With the same notations as in Conjecture 5.1, let $f_P(x)$, $f_{P_1}(x)$, $\dots$, $f_{P_n}(x)$ be the h-polynomials of $\mathscr{A}(P)$, $\mathscr{A}(P_1)$, $\dots$, $\mathscr{A}(P_n)$, respectively. Then,*

$$n!f_P(x) = \sum_{\lambda \vdash n} \frac{n!}{\ell(\lambda)!} R(\lambda) F_{(\lambda_1 - 1, \lambda_2 - 1, \dots)}(x) \left(\sum_{k=1}^{\ell(\lambda)} s_{\ell(\lambda),k} x^{\ell(\lambda)-k} f_{P_k}(x) \right), \tag{3}$$

where $R(\lambda)$ is the number of rearrangements of λ.

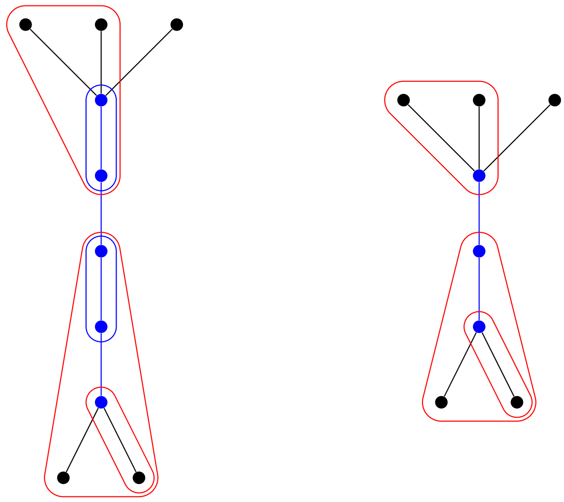


Figure 23: A degradable tubing of $Q(a \to C_5)$ (left) and a non-degradable tubing of $Q(a \to C_3)$ (right)

Proof. For each composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$ that is a rearrangement of a partition λ, the generating function for the degrading tubings of S whose composition is α is $F_{(\lambda_1-1,\lambda_2-1,\dots)}(x)$. This is because for each maximal tube τ of S, the tubes contained in τ form a tubing of $C_{|\tau|}$, and the generating function for such tubings is $F_{|\tau|-1}(x)$. By Lemma 5.7, degradable tubings of P in which the composition of the degrading tubings is α can be viewed as non-degradable tubings of $Q(a \to C_\ell)$. Then by Lemma 5.5, non-degradable tubings of $Q(a \to C_\ell)$ can be written as a sum of tubings of $P_1, \dots, P_\ell$ with coefficients $S_{\ell,k}$. This gives the desired formula. □

Proof of Proposition 5.4. For each partition λ, one can view λ as a tuple $(c_1, \dots, c_n)$ such that $\sum_i ic_i = n$. Then, in the RHS of (3),

$$R(\lambda) = \frac{\ell(\lambda)!}{c_1! \dots c_n!}.$$

Thus,

$$\frac{n!}{\ell(\lambda)!} R(\lambda) F_{(\lambda_1-1,\lambda_2-1,\dots)} = \frac{n!}{c_1! \dots c_n!} F_{(\lambda_1-1,\lambda_2-1,\dots)}(x)$$

$$= \frac{n!}{\lambda_1 \dots \lambda_\ell \cdot c_1! \dots c_n!} \lambda_1 \dots \lambda_\ell \cdot F_{(\lambda_1-1,\lambda_2-1,\dots)}(x) = \frac{n!}{\lambda_1 \dots \lambda_\ell \cdot c_1! \dots c_n!} \tilde{F}_\lambda(x).$$

Notice that $\frac{n!}{\lambda_1 \dots \lambda_\ell \cdot c_1! \dots c_n!}$ is the number of permutations in $\mathfrak{S}_n$ with cycle type λ, so the RHS of (3) becomes

$$\sum_{w \in \mathfrak{S}_n} \tilde{F}_w(x) \left(\sum_{k=1}^{\ell_w} s_{\ell_w,k} x^{\ell_w - k} f_{P_k}(x) \right).$$

Recall that $s_{n,k}$ is the coefficient of x^{n-k} in $1(1+x)\dots(1+(n-1)x)$. Hence, the coefficient of $f_{P_k}(x)$ in the above sum is the coefficient of t^k in

$$\sum_{w \in \mathfrak{S}_n} \tilde{F}_w(x) t(t+x) \dots (t + (\ell_w - 1)x),$$

which is the RHS of (2).

Finally, by the h-to-f-vector conversion, one can check that the coefficient of $f_{P_k}(x)$ in the RHS of (1) is the coefficient of t^k in the LHS of (2). Hence, Conjecture 5.2 implies Conjecture 5.1. □

5.2.1 Proof of Lemma 5.5

In order to prove Lemma 5.5, we will need a small bijection between

- pairs (w, α) where $w \in \mathfrak{S}_n$ and α is a composition of n into k parts, and
- pairs (ω, U) where ω is a permutation in $\mathfrak{S}_n$ with ℓ cycles and U is an ordered set partition of $\{1, \dots, \ell\}$ into k sets.

Our bijection is constructed as follows. Given a pair (w, α) where $w \in \mathfrak{S}_n$ and $\alpha = (\alpha_1, \dots, \alpha_k)$ is a composition of n into k parts:

1. Let $\mu_i = w_{\alpha_1+\ldots+\alpha_{i-1}+1} \dots w_{\alpha_1+\ldots+\alpha_i}$.
2. Let $V_i = \{v_1 < \ldots < v_{\alpha_i}\}$ be the set of elements in μ_i, then we can consider μ_i as a permutation of the elements $v_1, \dots, v_{\alpha_i}$. Let σ_i be the cycle decomposition of this permutation.
3. Let $\omega = \sigma_1 \dots \sigma_i$, this is the desired permutation.
4. Order the cycles in ω as $\nu_1, \dots, \nu_\ell$ in the order of their smallest element, then let $U_i = \{j \mid \sigma_i \text{ contains } \nu_j\}$. $(U_1, \dots, U_k)$ is the desired ordered set partition.

Example 5.9. Let $w = 965347128$ and $\alpha = (3, 4, 2)$.

1. $\mu_1 = 965, \mu_2 = 3471, \mu_3 = 28$.
2. $\sigma_1 = (59)(6)$ when we consider 965 as a permutation of 569; similarly, $\sigma_2 = (1347), \sigma_3 = (2)(8)$.
3. $\omega = (59)(6)(1347)(2)(8) = 324796185$.
4. The cycles are ordered as $\nu_1 = (1347), \nu_2 = (2), \nu_3 = (59), \nu_4 = (6), \nu_5 = (8)$, then $U_1 = \{3, 4\}$ since σ_1 contains ν_3 and ν_4; similarly, $U_2 = \{1\}, U_3 = \{2, 5\}$.

Proof of Lemma 5.5. We will construct a bijection between

- pairs (w, T) where $w \in \mathfrak{S}_n$ and T is a non-degradable tubing of $P = Q(a \to C_n)$ with k tubes, and
- pairs (ω, T') where ω is a permutation in $\mathfrak{S}_n$ with ℓ cycles and T' is a tubing of $P_\ell = Q(a \to A_\ell)$ with k tubes.

Our construction of T' from T follows the same idea as in Section 3.2. Recall that a tube $\tau \in T$ is *good* if $\tau \subseteq P - S$, $\tau \subseteq S$, or $S \subseteq \tau$ and is *bad* otherwise. We denote the set of good tubes by T_{good} and the set of bad tubes by T_{bad}. In this case, we do not have tubes $\tau \subseteq S$. Hence, we can keep T_{good} for T'. Then, we decompose T_{bad} into a triple $(\mathcal{L}, \mathcal{M}, \mathcal{U})$ where $\mathcal{L}$ and $\mathcal{U}$ are nested sequences of sets, some of which may be marked, contained in $P - S$ and $\mathcal{M}$ is an ordered set partition of $S = C_n$. Finally, we construct T'_{bad} from a triple $(\mathcal{L}, \mathcal{M}', \mathcal{U})$, where $\mathcal{M}'$ is an ordered set partition of some A_ℓ and $|\mathcal{M}'| = \mathcal{M}$, and have $T' = T_{\text{good}} \sqcup T'_{\text{bad}}$.

Hence, our bijection between (w, T) and (ω, T') comes down to a bijection between $(w, \mathcal{M})$ and $(\omega, \mathcal{M}')$, where $|\mathcal{M}'| = \mathcal{M}$.

Since $S = C_n$, there is an easy one-to-one correspondence between sequences $\mathcal{M}$ of S and compositions α of n. On the other hand, any ordered set partition of A_ℓ is an ordered set partition U of $\{1, \dots, \ell\}$. Therefore, a bijection between $(w, \mathcal{M})$ and $(\omega, \mathcal{M}')$, where $|\mathcal{M}'| = \mathcal{M}$, is essentially a bijection between (w, α) and (ω, U), which is the bijection discussed at the beginning of the section. $\square$

6 γ-positivity

6.1 Cyclic fence poset

Definition 6.1. The *(even) cyclic fence poset* $CF_{2(n+1)}$ is defined to be the poset on the elements $\{1, 2, \ldots, 2n+2\}$ with the covering relations $2k-1, 2k+1 \lessdot 2k$ for $1 \leq k \leq n$, and $1, 2n+1 \lessdot 2n+2$.

Similarly, the *(odd) cyclic fence poset* CF_{2n+1} is defined to be the poset on the elements $\{1, 2, \ldots, 2n+1\}$ with the covering relations $2k-1, 2k+1 \lessdot 2k$ for $1 \leq k \leq n$, and $1 \lessdot 2n+1$.

Example 6.2. Figure 24 gives examples of even and odd cyclic fence posets.

(a) CF_6 (b) CF_7

Figure 24: Cyclic fence posets

It turns out that the h-and-γ-vectors of cyclic fence posets have a particularly nice combinatorial interpretation in terms of colored paths.

Definition 6.3. A *colored* (m, n) *path* is a sequence of m upsteps and n downsteps where each step is colored red or blue. Let $CP_{m,n}$ denote the set of colored (m, n) path. A *peak* is an upstep followed by a downstep. A peak step is one of the two steps at some peak. The remaining steps are called *side steps*.

Example 6.4. Figure 25 shows a colored $(5, 4)$ path. This path has two peaks, three blue peak steps and one red peak step. The remaining five steps are side steps. Note that if the last step is an upstep, we do not consider it a peak step.

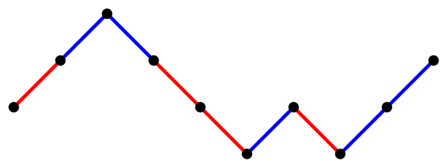


Figure 25: A path in $CP_{5,4}$

The following theorem relates colored paths and h-vectors of the poset associahedra of cyclic fence posets. The case for $CF_{2(n+1)}$ was found by Sack and appeared in his FPSAC 2023 Extended Abstract. The case for CF_{2n+1} was found later through private communication.

Theorem 6.5. *For $CF_{2(n+1)}$, the h-vector is given by*

$$h_i = |\{w \in CP_{n,n} \mid \#\textit{red peak steps} - \#\textit{blue peak steps} = 2(i-n)\}|.$$

For CF_{2n+1}, the h-vector is given by

$$h_i = |\{w \in CP_{n-1,n} \mid \#\textit{red side steps} - \#\textit{blue side steps} = 2(i-n)+1\}|.$$

Question 6.6. Our proofs for Theorem 6.5 use generating functions. It would be a nice problem to find a bijective proof for this theorem. In particular, the case for $CF_{2(n+1)}$ is related to Shapiro's convolution formula, which was proved by a complicated bijection in [HN14].

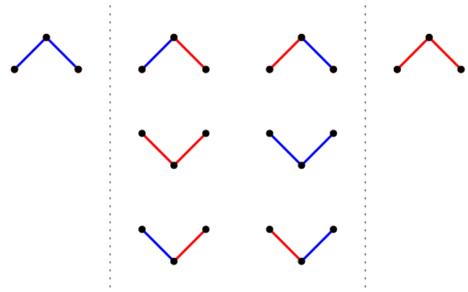


Figure 26: Paths in $CP_{1,1}$

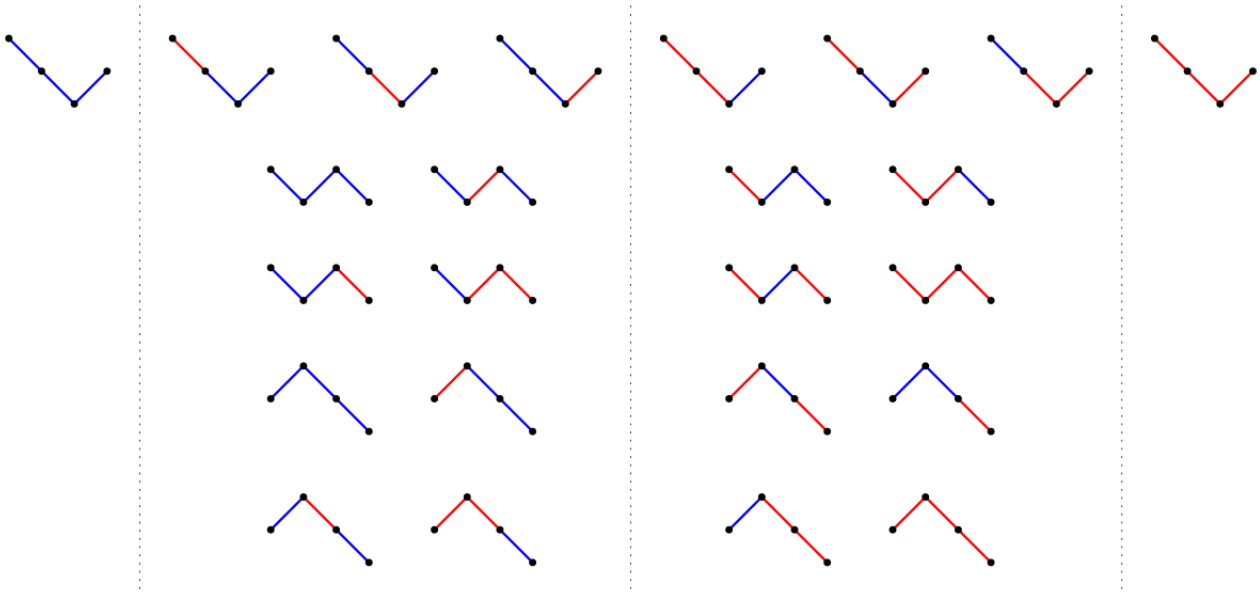


Figure 27: Paths in $CP_{1,2}$

Example 6.7. Figure 26 shows paths in $CP_{1,1}$. This means that the h-vector of CF_4 is $(1, 6, 1)$. Similarly, from Figure 27, the h-vector of CF_5 is $(1, 11, 11, 1)$.

Corollary 6.8. *For $CF_{2(n+1)}$, the γ-vector is given by*

$$\gamma_i = 4^i \binom{n}{i}^2.$$

For CF_{2n+1}, the γ-vector is given by

$$\gamma_i = 4^i \binom{n}{i} \binom{n-1}{i}.$$

In particular, the poset associahedra of cyclic fence posets are γ-positive.

Proof. For $CF_{2(n+1)}$, let P be a path with n upsteps, n downsteps, $n-i$ peaks, and all side steps colored red or blue. Then, the 2^{n-i} coloring of the peak steps of P contribute $x^i(x+1)^{2n-2i}$ to the h-polynomial $h_{CF_{2(n+1)}}(x)$.

Observe also that the number of such paths P is $4^i\binom{n}{i}^2$. This is because the number of path with n upsteps, n downsteps, and $n-i$ peaks is $\binom{n}{i}^2$ (see [Sim03, Proposition 2]). Each of the $2i$ side steps can be colored either red or blue, so there are 4^i ways to color each path. This gives the desired formula.

The case for CF_{2n+1} is similar. $\square$

6.2 γ-positivity conjectures

A simplicial complex Δ is a *flag complex* if its simplices are exactly the cliques of some graph. A simple polytope is *flag* if its dual simplicial complex is flag.

In [Gal05], Gal conjectured that every flag simple polytope is γ-nonnegative. This conjecture has been proved for several family of polytopes. For example, Postnikov–Reiner–Williams ([PRW06]) proved the conjecture for nestohedra of connected chordal building sets, which include graph associahedra of chordal graphs (e.g. trees). Volodin ([Vol10]) proved the conjecture for the class $sd(\Sigma_{d-1})$ of simplicial complexes that can be obtained from Σ_{d-1} by stellar subdivisions in edges. See also: [Ais12, Ath12, Ero09, Gor11].

Not all poset associahedra are flag. In fact, the minimal example of non-flag poset associahedra is the cyclic fence poset CF_6. However, Corollary 6.8 shows that the poset associahedra of cyclic fence posets are still γ-positive. Thus, we make the following conjecture.

Conjecture 6.9. *All poset associahedra are γ-positive.*

Computational evidence suggested a stronger evidence. Let C_n be poset that is a chain of n elements, and $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ be the complete bipartite poset $A_{\lfloor \frac{n}{2} \rfloor} \oplus A_{\lceil \frac{n}{2} \rceil}$. Note that C_n is the poset on n elements with the fewest covering relations, and $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is the poset with the most covering relations. In addition, we say $(a_1, \ldots, a_k) \lll (b_1, \ldots, b_k)$ if $a_i < b_i$ for all $1 \leq i \leq k$. We have the following conjecture.

Conjecture 6.10. *Let P be a connected poset on n elements, and P is not C_n or $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Then the h-and-γ-vectors of $\mathscr{A}(P)$ satisfy*

$$h_{C_n} \lll h_P \lll h_{K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}},$$
$$\gamma_{C_n} \lll \gamma_P \lll \gamma_{K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}}.$$

Conjecture 6.10 has been checked for all connected poset of size up to 7. In fact, we have an even stronger conjecture.

Conjecture 6.11. *The h-polynomials of poset associahedra are real-rooted.*

Remark 6.12. Despite Conjecture 6.10, there is no apparent relationship between the number of covering relations of a poset and the face numbers of its poset associahedra. For example, the posets $A_2 \oplus A_2$ and $A_1 \oplus A_2 \oplus A_1$ both have four covering relations, but their face numbers are different.

Question 6.13. In [PRW06], Postnikov–Reiner–Williams proved Gal's conjecture for nestohedra $P_{\mathcal{B}}$ of connected chordal building sets $\mathcal{B}$ by finding a set of permutations $\mathfrak{S}(\mathcal{B})$ such that the descent-generating function of $\mathfrak{S}(\mathcal{B})$ and $\hat{\mathfrak{S}}(\mathcal{B})$ gives the h-and-γ-vectors of $P_{\mathcal{B}}$, respectively. Here $\hat{\mathfrak{S}}(\mathcal{B})$ is the set of permutations in $\mathfrak{S}(\mathcal{B})$ with no double descents or final descent. They asked whether such $\mathfrak{S}(\mathcal{B})$ exists for any building set $\mathcal{B}$ [PRW06, Question 14.3]. This question was answered partly in [Ero09] for connected building sets. Here we ask a similar question for poset associahedra, that is: for any connected poset P, is there a set of permutations $\mathfrak{S}(P)$ such that the descent-generating function of $\mathfrak{S}(P)$ and $\hat{\mathfrak{S}}(P)$ gives the h-and-γ-vectors of $\mathscr{A}(P)$?

In fact, it remains an open problem to find a combinatorial interpretation for the face numbers of poset associahedra.

Question 6.14. Find a combinatorial interpretation for the face numbers of poset associahedra.

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