Modeling Interest Rates

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Recent Interest Rate Research (after a 15 year break) See www-2.rotman.utoronto.ca/~hull


Our 1990s Approach

Assume that

\[ dx = [\theta(t) - ax]dt + \sigma dz \]

where \( x \) is some function of the short rate \( r \)

- \( x = r \) and \( a = 0 \) gives Ho-Lee (1986)
- \( x = r \) and \( a > 0 \) gives Hull-White (1990)
- \( x = \ln(r) \) and \( a = 0 \) gives Kalotay-Williams-Fabozzi (1993)
- \( x = \ln(r) \) and \( a > 0 \) gives Black-Karasinski (1991)
Building the 1990s Tree

- Define a variable $x^*$ which follows
  \[ dx^* = -ax^* dt + \sigma dz. \]
- Construct a trinomial tree for $x^*$
- Set $x = x^* + \phi(t)$ and choose $\phi(t)$ to match the term structure
- This leads to
  \[ dx = [\theta(t) - ax] dt + \sigma dz. \]
  where
  \[ \theta(t) = \phi'(t) + a\phi(t). \]
Building the 1990s Tree continued
The Problem

This approach only works for models of the form

\[ dx = [\theta(t) - ax]dt + \sigma dz \]

where \( x = f(r) \)

It does not allow users to independently specify the drift and volatility of \( r \)

In Black-Karasinski the process for \( r \) is

\[ dr = r \left( (\theta(t) - a \ln(r)) + \frac{\sigma^2}{2} \right) dt + \sigma r dz \]
Why we need a new approach in today’s low interest rate environment

- Normal model does not work well. (Probability of negative rates too high)
- Lognormal model does not work well (Assumes rate increase from 20bp to 40bp has same probability as an increase from 10% to 20%)
- Volatility of $r$ is not a simple function of $r$
...concludes from historical data that short rate is approximately lognormal for low rates and high rates and normal for intermediate rates.
New Approach

Suppose

\[ dr = \left[ \theta(t) + F(r) \right] dt + G(r) \, dz \]

Define

\[ x = f(r) = \int \frac{dr}{G(r)} \]

so that

\[ dx = H(x, t) \, dt + dz \]

where

\[ H(x, t) = \frac{\theta(t) + F(r)}{G(r)} - \frac{1}{2} G'(r) \]
The Tree

\[ \Delta x = \sqrt{3\Delta t} \]
Tree Building

- At each time \((i-1)\Delta t\) we choose a trial value for \(\theta(t)\). For \(j\)th node at this time:
  - Calculate a drift \(m_{i-1,j}\) (being careful to ensure that it is correct in the limit as \(\Delta t\) tends to zero)
  - Choose node at time \(i\Delta t\) closest to \(x_{i-1,j} + m_{i-1,j} \Delta t\) and branch to the triplet of nodes centred on this node
  - Choose branch probabilities to match first two moments of change in \(x\)
  
- Value an \((i+1)\Delta t\) zero-coupon bond
  
- Iterate until bond price matches that calculated from the term structure
Very occasionally branching oscillates and term structure cannot be matched. When this is observed to happen, we freeze the branching process at \((i-1)\Delta t\) and then proceed to match the term structure.

To calculate Greek letters we freeze branching process.

We sometimes need to adjust the assumed drift of \(r\) so that it never becomes negative as \(r\) tends to zero.
Trial Volatility Function, $G(r)$

Drift is $a[\theta(t) - r]$
Convergence (Annual pay caps using Dec 2, 2013 term structure)

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Determining the Market Implied Volatility Function

- Assume that $G(r)$ is piecewise linear
- Develop a procedure for “rounding the corners”
- Use a “goodness of fit” objective function to determine how well market prices are being matched:

$$\sum_{i=1}^{N} \frac{(U_i - V_i)^2}{U_i}$$

$U_i$ and $V_i$ and market price and model price, respectively
First Test
Second Test  (Red line is average)
Second test continued (red line is average)
Comparison with Historical Results

Girsanov’s theorem states that volatility function should be the same for the real world and the risk-neutral world.

Our results provide evidence that market participants are using a volatility function similar to that derived by Deguillaume, Rebonato, and Pogudin for the real world.
From Q- to P-measure

If the Q-measure process backed out from prices is

\[ dr = \mu(r,t)dt + \sigma(r)dz \]

the P-measure (real-world) process is

\[ dr = [\mu(r,t) + \lambda\sigma(r)]dt + \sigma(r)dz \]

where \( \lambda \) is the market price of risk
The Value of $\lambda$

- When estimated from short rates $\lambda$ is between -0.5 and -2.5 (see for example Stanton (1997), Cox and Pedersen (1999) and Ahmad and Wilmott (2007))
- If we use a value of, say, -1.0 in conjunction with a reasonable amount of mean reversion we find that we find that the expected real world short rate in 20 or 30 years is unreasonably low (and for some models very negative)
Estimates of $\lambda$ decline when longer rates are used.
Why Do Different Approaches Give Different Estimates for \( \lambda \)

- Interest rates follow a multifactor process
- Each factor has its own market price of risk
- Estimating \( \lambda \) assuming only one factor gives strange results when used to estimate expected future short rate
We introduce the concept of a local price of risk so that $\lambda$ is a function of time.

The $\lambda$ function can be chosen so that:
- it reflects the average slope of the term structure at each maturity; or
- it is consistent with the average short rate observed in the past (about 4.4%).
Using Tree to Value Instrument Dependent on 3-month LIBOR with OIS Discounting

- Calculate OIS and LIBOR zero curves
- Assume a process for the instantaneous OIS rate and construct a tree for the $\Delta t$ OIS rate
- Roll back to calculate the 3-month OIS rate at each node
- Assume a one factor model for the spread between 3-month OIS and 3-month LIBOR. This will include a function of time in the drift.
Trees for LIBOR and OIS continued

- Assume a correlation between three-month OIS and spread
- Work forward calculating the expected spread at each time to match forward LIBOR. (The unconditional expected spread at a particular time determines the conditional spread at each node at that time from the correlation assumptions)
- We then have a tree where there is a $\Delta t$ OIS rate and an expected 3-month LIBOR rate at each node.
Conclusions

- We have developed a new procedure for building trees for one-factor interest rate models. This can accommodate a much wider range of model assumptions that those underlying the 1990s tree building procedures.

- Results going back to 2004 provide support for the volatility function determined from historical data by Deguillaume, Rebonato, and Pogudin.

- One factor trees can be used to a) determine the P-measure and b) accommodate OIS discounting.