The Coupling Constant Metamorphosis and Superintegrable Systems

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Goals

- To give background on Coupling Constant Metamorphosis (CCM) also known as the Stäckel transform and its application to integrable systems.
- To show how Willard Miller and collaborators applied the concept to great results in the classification of second-order superintegrable systems.
- To show how Willard Miller and collaborators extended transform to higher order integrals of the motion, especially for quantum systems.
- To describe some recent applications to higher order superintegrable systems.
Outline

1. Background

2. Separation of Variables
   - Second Order Superintegrability and CCM

3. Special Functions
   - Representations of the Symmetry Algebras

4. Superintegrability
   - The extention of CCM to Nth order superintegrable systems
   - CCM between Deformed Coulomb and the TTW systems
Superintegrable Systems

In $n$ dimensions, we call a classical or quantum Hamiltonian

$$\mathcal{H} = \sum_{i=0}^{n} p_i^2 + V(x_i), \quad H = \Delta + V(x_i)$$

(maximally, Ntabsh-order) **Superintegrable** if it admits $2n - 1$ symmetry operators, ie.

$$\{L_i, \mathcal{H}\} = 0, \quad [L_i, H] = 0, \quad \forall i = 0, \ldots, 2n - 1$$

which are polynomial in the momenta or as differential operators.

We also require that these operators be independent in some sense.

**Superintegrable systems can be solved algebraically as well as analytically and are associated with special functions and exact solvability.**
Associated Symmetry Algebras

The operators $\mathcal{L}_i$ cannot all commute with each other we have non-trivial algebra relations

$$\{\mathcal{L}_i, \mathcal{L}_j\} = \mathcal{R}_{ij}$$

where $\mathcal{R}_{ij}$ are (usually new) symmetries of the Hamiltonian.

Often, if we include these new symmetries in our algebra, our algebra will close, i.e.

$$\{\mathcal{L}_i, \mathcal{R}_{ij}\} = P(\mathcal{L}_i), \quad P \text{ is a polynomial.}$$

Since $2n - 1$ is maximal number of functionally independent classical symmetries, there must be a functional relation between the $\mathcal{R}_{ij}$'s the $\mathcal{L}_i$'s.

These relations determine our symmetry algebra.
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**These relations determine our symmetry algebra.**
Given a Hamiltonian system in a $2n$ dimensional phase space

$$\mathcal{H}(q, p) + \tilde{E}U(q) = E$$

$$\iff \mathcal{H}' \equiv \frac{\mathcal{H}(q, p) - E}{U(q)} = \tilde{E}$$

We have essentially solved the Hamilton-Jacobi equations for $\tilde{E}$
Classical Coupling Constant Metamorphosis (CCM)

Given a Hamiltonian system $n \times 2n$ dimensional phase space

$$\mathcal{H}(q, p) + \tilde{E} U(q) = E$$

CCM preserves integrals of the motion:


Given a classical Hamiltonian $\mathcal{H} = \hat{H} - \tilde{E} U$, where $\hat{H}$ is independent of the arbitrary parameter $\tilde{E}$, with an integral of the motion $\mathcal{L}(\tilde{E})$. If we define the Stäckel transform of $\mathcal{H}$ and $\mathcal{L}$ as $\tilde{\mathcal{H}} \equiv U^{-1}(\hat{H} - E)$ and $\tilde{\mathcal{L}} \equiv \mathcal{L}(\tilde{H})$ then $\tilde{\mathcal{L}}$ is an integral of the motion for $\tilde{\mathcal{H}}$. .
Proof

The key to the proof is the following identity for Poisson brackets

\[
\{ F(p_i, q_j, f(p_i, q_j)), G \} = \{ F(p_i, q_j, \tau), G \} \bigg|_{\tau=f(p_i,q_j)} + \frac{\partial F(p_i, q_j, \tau)}{\partial \tau} \bigg|_{\tau=f(p_i,q_j)} \{ f(p_i, q_j), G \}
\]

where \( p_i, q_j \) are the conjugate position and momenta and \( \tau \) is a parameter. With this, we compute,

\[
\{ \tilde{H}, \tilde{L} \} = \left\{ \frac{1}{U}(H + \tilde{E}U - E), \mathcal{L} \right\}_{\tilde{E}=\tilde{H}}
\]

\[
= \left\{ \frac{1}{U}(H + \tilde{E}U - E), \mathcal{L} \right\}_{\tilde{E}=\tilde{H}} + \partial_{\tilde{E}} \mathcal{L}(\tilde{E})\bigg|_{\tilde{E}=\tilde{H}} \{ \tilde{H}, \tilde{H} \}
\]

\[
= -\{ U, \mathcal{L} \} \frac{1}{U^2} (H - E)\bigg|_{\tilde{E}=\tilde{H}}
\]

and since \( H\big|_{\tilde{E}=\tilde{H}} = E \), we see that \( \{ \tilde{H}, \tilde{L} \} = 0 \) and we have proved the theorem.
In general, the constant of the motion need not be polynomial in $\tilde{E}$ and so the transformed constant might not be polynomial in the momenta.

**Example (Gravel (J. Math. Phys., 45: 1003-1019, 2004))**

$$\mathcal{H} = p_1^2 + p_2^2 + b_1 \sqrt{x_1} + b_2 x_2$$

admits $\mathcal{L}^{(2)} = p_2^2 + b_2 x_2 \quad \mathcal{L}^{(3)} = p_1^3 + \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3b_1^2}{4b_2} p_2$
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The choice $\tilde{E} = b_2$ leads to symmetry a symmetry operator which is a rational function of the momenta.
Example

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Transformed System

\[ \tilde{\mathcal{H}} = \frac{1}{x_2} (p_1^2 + p_2^2 + b_1 \sqrt{x_1} - E) \]
\[ \mathcal{L}^{(2)} = p_2^2 + p_1^2 + p_2^2 + b_1 \sqrt{x_1} - E \]
\[ \mathcal{L}^{(3)} = p_1^3 + \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3 b_1^2 x_2}{4(p_1^2 + p_2^2 + b_1 \sqrt{x_1} - E)} p_2 \]
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\[ \tilde{\mathcal{H}} = \left( \frac{1}{x_2} \right) (p_1^2 + p_2^2 + b_1 \sqrt{x_1} - E) \]

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\[ \mathcal{L}^{(3)} = p_1^3 + \frac{3}{2} b_1 \sqrt{x_1} p_1 - \frac{3 b_1^2 x_2}{4(p_1^2 + p_2^2 + b_1 \sqrt{x_1} - E)} p_2 \]

Note that the metric of the system has changed

\[ H = g^{ij} p_i p_j + V, \quad H' = g^{ij} p_i p_j + U \]

admit separation of variables in the same orthogonal coordinate system then the Hamiltonian given by

\[ \tilde{H} = U^{-1} \left( g^{ij} p_i p_j + V \right) \]

will also admit separation of variables in the same coordinates.

This transform was called the "Stäckel Transform" because it preserves the Stäckel form of the separable systems. The transform applies equally to classical and quantum systems.
Classification of Second Order Superintegrable Systems

The relation between CCM and the Stäckel transform becomes apparent when applied to second-order superintegrable systems. In a series of papers beginning in 2005 Miller and collaborators, Kalnins Kress Pogosyan, proved the classification of second order superintegrable systems.

- Besides a special degenerate case, all such systems are separable in multiple coordinate systems determined by the 2 independent integrals of the motion.
- The potentials depend on at most 4 independent parameters and are linearly dependent on them.
Stäckel Transform of Second-Order Superintegrable systems

We consider non-degenerate second-order superintegrable systems in 2D,

\[ H = p_1^2 + p_2^2 + \alpha V_\alpha + \beta V_\beta + \gamma V_\gamma + \delta \]

with symmetry operators \( L_1(\alpha, \beta, \gamma), L_2(\alpha, \beta, \gamma) \) also depend linearly on the constants \( \alpha, \beta, \gamma \). We can then use any or all of the potential to preform the Stäckel transform. For example, take as above

\[ H' = p_1^2 + p_2^2 + V_\alpha, \quad L'_i(1,0,0) \]

\[ \tilde{H} = (V_\alpha)^{-1} H - \alpha \]

will be a new Hamiltonian which is also separable in the same orthogonal coordinate systems.
We can also write the symmetry operators

\[ L_i(\alpha, \beta, \gamma) = a^{ij} p_ip_j + \alpha W_\alpha + \beta W_\beta + \gamma W_\gamma \]

Then, the Stäckel transform requires that the symmetry operators be mapped to

\[ \tilde{L}_i = a^{ij} p_ip_j + \alpha W_\alpha + \beta W_\beta + \gamma W_\gamma - W_\alpha U^{-1}H \]

\[ = a^{ij} p_ip_j - \tilde{H} W_\alpha + \beta W_\beta + \gamma W_\gamma \]

The Stäckel transform and CCM differ by a trivial additive constant.
Classification

For any second-order superintegrable system in 2D, we can always divide through by the entire potential so that the classification theory of superintegrable systems reduces to the problem treated by Koenigs (Lecons sur la théorie générale des surfaces, 1872) of identifying metrics that admit three (including the Hamiltonian) independent second-order Killing tensors.

Furthermore, Miller et al. showed that any second-order superintegrable system can be mapped by the Stäckel transform to one on a space of constant curvature. Thus, the problem of classifying such potentials was significantly reduced.
Example: Simple Harmonic Oscillator and Coulomb Potential

The harmonic oscillator and the Kepler system are related by an example of CCM which maps between systems which are both on the plane.
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The harmonic oscillator and the Kepler system are related by an example of CCM which maps between systems which are both on the plane.

\[-\partial_x^2 - \partial_y^2 + \tilde{E}(x^2 + y^2) = E\]
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Solving for \( \tilde{E} \)

\[ \frac{1}{x^2 + y^2} \left( \partial_x^2 + \partial_y^2 - E \right) = \tilde{E} \]

\( \tilde{H} = \frac{H - E}{U} \)
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\[ \frac{1}{x^2 + y^2} \left( \partial_x^2 + \partial_y^2 - E \right) = \tilde{E} \]

Using the change of variables \( \nu = \frac{1}{4}(x^2 - y^2), \quad \zeta = xy \) we obtain

\[ \tilde{H} = \partial_{\nu}^2 + \partial_{\zeta}^2 - \frac{E}{2\sqrt{\nu^2 + \zeta^2}}. \]
Example cont.

The symmetry operators for the SHO are

\[ X = M = x \partial_y - y \partial_x \]

\[ L_1 = \frac{1}{2} (M \partial_x + \partial_x M) - \frac{\alpha y}{2 \sqrt{x^2 + y^2}}, \quad L_2 = \frac{1}{2} (M \partial_y + \partial_y M) - \frac{\alpha x}{2 \sqrt{x^2 + y^2}}. \]

The transformed constants are

\[ \tilde{X} = M = x \partial_y - y \partial_x \]

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The symmetry algebra for the SHO is

\[ [L_1, X] = L_2 \quad [L_2, X] = -L_1 \quad [L_1, L_2] = -\alpha X \]

\[ L_1^2 + L_2^2 + \alpha X^2 - \frac{\alpha}{4} - \frac{E^2}{16} = 0. \]

The symmetry algebra for the Coulomb system

\[ [\tilde{L}_1, \tilde{X}] = \tilde{L}_2 \quad [\tilde{L}_2, \tilde{X}] = -\tilde{L}_1 \quad [\tilde{L}_1, \tilde{L}_2] = \tilde{E} \tilde{X} \]

\[ \tilde{L}_1^2 + \tilde{L}_2^2 - \tilde{E} \tilde{X}^2 + \frac{\tilde{E}}{4} - \frac{\tilde{\alpha}^2}{4} = 0. \]

These are identical under the identifications

\[ \tilde{L}_1 = L_1, \quad \tilde{L}_2 = L_2, \quad \tilde{E} = -\alpha, \quad \tilde{\alpha} = -\frac{1}{2} E. \]

We can use the Stäckel transform to understand symmetry algebras that differ by a permutation of the structure constants!
Example cont.

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CCM and the symmetry algebra

Q: So the question is, can we determine when two such systems are related by the CCM?

- Because the effect of CCM on the symmetry algebra is simply to intertwine the energy with a parameter, we can immediately see its effect.

- In the case of second-order superintegrable systems and their quadratic algebras we can see explicitly the dependence of the algebra on the parameters and the energy and so show that the CCM will map such quadratic algebras to new quadratic algebras and will leave invariant the highest order terms in the "Casimir" relation.
Further Classifications

Because of the invariance of the symmetry algebra under CCM, we can classify equivalence classes of superintegrable systems. This work was completed for 2D second-order superintegrable systems by Kress (Phys. Atomic Nuclei 17:560-566, 2007).

<table>
<thead>
<tr>
<th>Leading terms of Casimir relation</th>
<th>system</th>
<th>Operator model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1^3 + f(\alpha_i, H)L_2^2$</td>
<td>E2, S1</td>
<td>Differential</td>
</tr>
<tr>
<td>$L_1^3 + f(\alpha_i, H)L_1L_2$</td>
<td>E9, E10</td>
<td>Differential</td>
</tr>
<tr>
<td>$L_1^3 + 0$</td>
<td>E15</td>
<td>Differential</td>
</tr>
<tr>
<td>$L_1^2L_2 + f(\alpha_i, H)L_2^2$</td>
<td>E1, E16,S2, S4</td>
<td>Differential</td>
</tr>
<tr>
<td>$L_1^2L_2 + 0$</td>
<td>E7, E8, E17, E19</td>
<td>Differential</td>
</tr>
<tr>
<td>$L_1L_2(L_1 + L_2) + f(\alpha_i, H)L_1L_2$</td>
<td>S7, S8, S9</td>
<td>Difference</td>
</tr>
<tr>
<td>$0 + f(\alpha_i, H)L_1L_2$</td>
<td>E3, E11, E20</td>
<td>Differential</td>
</tr>
</tbody>
</table>

Table: Stäckel equivalence classes of non-degenerate systems in 2d
Why Equivalence classes?

The representations of the algebras are isomorphic on Stäckel equivalence classes.

Example (S9)

The Hamiltonian is

\[ H = \Delta_{S^2} + \frac{1}{4} - a^2 \frac{s_1^2}{s_2^2} + \frac{1}{4} - b^2 \frac{s_2^2}{s_3^2} + \frac{1}{4} - c^2 \frac{s_3^2}{s_1^2}. \]

A basis for the symmetry operators is,

\[ L_1 = J_3^2 + \left( \frac{1}{4} - a^2 \right) \frac{s_1^2}{s_2^2} + \left( \frac{1}{4} - c^2 \right) \frac{s_2^2}{s_1^2}, \]

\[ L_2 = J_1^2 + \left( \frac{1}{4} - a^2 \right) \frac{s_3^2}{s_2^2} + \left( \frac{1}{4} - b^2 \right) \frac{s_2^2}{s_3^2}, \]

\[ H = L_1 + L_2 + L_3 + \frac{3}{4} - a^2 - b^2 - c^2. \]
The structure equations can be put in the symmetric form using the following identification,

\[ a_1 = \frac{1}{4} - c^2, \quad a_2 = \frac{1}{4} - a^2, \quad a_3 = \frac{1}{4} - b^2. \]

\[ [L_i, R] = \epsilon_{ijk} \left( 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - \right. \]

\[ R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 \]

\[ + \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 \]

\[ + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) \]

\[ + 48(a_1a_2 + a_2a_3 + a_3a_1) + 64a_1a_2a_3. \]
Kalnins, Miller and P. (J. Math Phys. A, 40: 11525-11538, 2007) found a representation of this system using difference operators. A suitable model diagonalizing $L_3$ is realized by setting

$$L_3 = -4t^2 + a^2 + c^2$$

We can obtain $L_1$ in the model by using the following difference operator, based upon the Wilson polynomial algebra. We simplify the model by using the substitutions

$$\alpha = -\frac{a + c + 1}{2} - m, \quad \beta = \frac{a + c + 1}{2}, \quad \gamma = \frac{a - c + 1}{2}, \quad \delta = \frac{a + c - 1}{2} + b + m$$

$$T^A F(t) = F(t + A), \quad \tau = \frac{1}{2t} (T^{1/2} - T^{-1/2}),$$

$$\tau^* = \frac{1}{2t} \left[ (\alpha + t)(\beta + t)(\gamma + t)(\delta + t) T^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t) T^{-1} \right]$$

to express the action of $L_1$:

$$L_1 = -4\tau^* \tau - 2(a + 1)(b + 1) + \frac{1}{2}.$$
Quantum CCM

We now consider superintegrable systems with higher order integrals of the motion. To apply CCM to such systems we must make some assumptions. The following theorem was proven, Kalnins, Miller and P. (J. Phys. A. 43: 035202, 2010).

Theorem

Given a quantum Hamiltonian $H = H_0 - \tilde{E} U$, where $H_0$ is independent of the arbitrary parameter $\tilde{E}$, with an integral of the motion

$L = \sum_{j=0}^{\left[\frac{N}{2}\right]} K_{N-2j} \tilde{E}^j$, where $K_i$ have degree $i$ as differential operators. If we define the Stäckel transform of $H$ and $L$ as $\tilde{H} = U^{-1}(H_0 - E)$ and $\tilde{L} = \sum_{j=0}^{\left[\frac{N}{2}\right]} K_{N-2j} \tilde{H}^j$, then $[\tilde{H}, \tilde{L}] = 0$.

Furthermore, if $H$ is self-adjoint and $L$ is self or skew adjoint, depending on the parity of $N$, with respect to $d_\mu$ then $\tilde{H}$ will be self-adjoint and $\tilde{L}$ will have the same parity as $L$ with respect to the metric $U d_\mu$. 
Proof

We use the independence of the parameter $\tilde{E}$ to show $[\tilde{H}, \tilde{L}] = 0$. then we have

$\left[ L, H \right] = [\sum_{j=0}^{\lfloor N/2 \rfloor} K_{N-2j} \tilde{E}^j, H_0 + \tilde{E} U] = 0 \iff [K_{N-2j}, H_0] + [K_{N-2j+2}, U] = 0$

We then use $[K_{N-2j}, \tilde{H}] = U^{-1}[K_{N-2j}, H_0] - U^{-1}[K_{N-2j}, U] \tilde{H}$ to compute,

$[\tilde{L}, \tilde{H}] = \sum_j \left( [K_{N-2j}, H_0] + [K_{N-2j+2}, U] \right) \tilde{H}^j = 0.$

Similarly

$\int Lfgd\mu = (-1)^N \int fLgd\mu \iff \int K_{N-2j}fgd\mu = (-1)^N \int fK_{N-2j}gd\mu$

$\Rightarrow \int \sum_i K_{N-2j} \tilde{H}^i fg Ud\mu = \sum_i (-1)^N \int f(\tilde{H}^i U^{-1} K_{N-2j} Ug) Ud\mu$
The Potential of Tremblay, Turbiner and Winternitz (TTW)

In (J. Phys. A, 42:242001, 2009) Tremblay, Turbiner and Winternitz introduced a family of deformations of the simple harmonic oscillator indexed by a parameter $k$ and defined by the potential

$$V = \omega^2 \rho^2 + \frac{\alpha k^2}{\rho^2 \cos^2 (k \theta)} + \frac{\beta k^2}{\rho^2 \sin^2 (k \theta)}$$

where $(\rho, \theta)$ are polar coordinates on $\mathbb{R}^2$. In two papers, the authors proved

- The bounded trajectories of the classical system are closed for rational $k$.
- The quantum system is exactly solvable and has eigenvectors which are gauge equivalent to Jacobi multiplied by Laguerre polynomials, for integer $k$.
- The system is integrable for all values of $k$ and superintegrable for $k = 1, 2, 3, 4$. 
Separability

The system is separable in polar coordinates and admits the integrals of motion

\[
L_{1 \text{TTW}} = p_{\theta}^2 + \frac{\alpha k^2}{r^2 \cos^2(k \theta)} + \frac{\beta k^2}{r^2 \sin^2(k \theta)}
\]

\[
L_{1 \text{TTW}} = -\partial_{\theta}^2 + \frac{\alpha k^2}{r^2 \cos^2(k \theta)} + \frac{\beta k^2}{\sin^2(k \theta)}
\]

The system for \( k = 1 \) also admits separation in cartesian coordinates defined by

\[
L_{2 \text{TTW}} = p_x^2 + \omega^2 x^2 + \frac{\alpha}{x^2}
\]

\[
L_{2 \text{TTW}} = \partial_x^2 + \omega^2 x^2 + \frac{\alpha}{x^2}
\]

and so is second-order superintegrable.
Bounded Trajectories

Trajectories are bounded under the following restrictions

\[ A > 0, \quad \alpha > 0, \quad \beta > 0, \omega > 0, \quad A > k^2|\beta - \alpha| \]

\[ \left( A - (\alpha + \beta)^2 k^2 \right)^2 - 4\alpha\beta k^2 > 0, \quad E^2 - 4\omega^2 A > 0 \]

The following trajectories are closed for rational \( k \)

\[ \rho^2 = \frac{1}{2\omega^2} \left( E + \sqrt{E^2 - 4\omega^2 A} \sin[4\omega(t + \delta_1)] \right) \]

\[ k \arcsin R + \arcsin U_k = -4k\sqrt{A}\delta_2 \]

\[ R = \frac{-2A + E\rho^2}{\rho^2\sqrt{E^2 - 4\omega^2 A}} \quad \quad U_k = \frac{-2A\sin^2 k\theta + A - (\alpha - \beta)k^2}{\sqrt{(A - (\alpha + \beta)k^2)^2 - 4\alpha\beta k^2}} \]
Wave functions

Solving the time independent Schrödinger equation

\[ H^{TTW} \Phi - E \Phi = 0 \]

we can separate variables and obtain the wave function as a Laguerre polynomial times a Jacobi polynomial

\[ \Phi = \rho^{2nk} L_N^{k(2n+\alpha+\beta)}(\omega \rho^2) P_n^{(a-1/2,b-1/2)}(2 \sin^2 k \theta - 1), \]

\[ G = \rho^{(a+b)k} \cos^a k \theta \sin^b k \theta e^{-\omega r^2} \]

the gauge, and quantized energy \( E = 2\omega (2N + (2n + a + b)k + 1) \)

and

\[ \alpha = a(a - 1) > -\frac{1}{4}, \quad \beta = b(b - 1) > -\frac{1}{4} \]
Symmetry algebras

For this system, there are two different types of symmetry algebras present

- A dynamic or "hidden" algebra whose enveloping algebra contains the Hamiltonian as well as the integrals of motion.

- A polynomial algebra generated by the integrals of the motion which form a subalgebra of the above algebra. We hypothesize that with the addition of $R \equiv [L_1, L_2]$ the algebra closes.
A New Family

In analogy with the TTW system, we consider a family of deformations of the Coulomb potential, introduced by P. and Winternitz (J. Phys. A 43:222001, 2010),

\[ V^{DC}_k = -\frac{Q}{r} + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2}\phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2}\phi\right)}. \]

In the paper, the following was proven

- The bounded trajectories of the classical system are closed for rational \( k \).
- The quantum system is exactly solvable and has eigenvectors which are gauge equivalent to Jacobi multiplied by Laguerre polynomials.
- The system is equivalent to the TTW system by Coupling Constant Metamorphosis (CCM).
Classical Trajectories

We look for bounded trajectories which satisfy the Hamilton-Jacobi equations $\mathcal{H} - E = 0$ under the restrictions

$$Q > 0, \quad \beta > 0, \quad \alpha > 0, \quad A > 0, \quad E > 0, \quad 4A - k^2|\beta - \alpha| > 0,$$

$$Q^2 + 4AE > 0 \quad (A - \frac{k^2}{4}(\beta + \alpha))^2 - \frac{\alpha \beta k^4}{4} > 0$$

$$0 = \frac{-2Er - Q}{\sqrt{Q^2 + 4AE}} + \sin \left( \frac{4(-E)^{3/2}}{Q}(t + \delta_1) + \frac{2\sqrt{-E}}{a} \sqrt{Er^2 + Qr - A} \right)$$

$$\delta_2 = \frac{1}{2\sqrt{A}} \arcsin(R) + \frac{1}{2k\sqrt{A}} \arcsin(U_k).$$

$$R = \frac{2A - Qr}{r\sqrt{Q^2 + 4AE}}, \quad U_k = \frac{2A\sin^2\left(\frac{k}{2}\phi\right) - A + \frac{k^2}{4}(\beta - \alpha)}{\sqrt{(A - \frac{k^2}{4}(\beta + \alpha))^2 - \frac{\alpha \beta k^4}{4}}}$$

These bounded trajectories are periodic in $t$ and $\phi$, for rational $k$. 
Quantum Eigenfunctions

The solutions for the Schrödinger equation \((H - E)\psi = 0\) are

\[
\psi = G_1 G_2 L_n^{2\sqrt{A}} \left(2r\sqrt{-E}\right) P_m^{a-\frac{1}{2},b-\frac{1}{2}} (-\cos(k\phi))
\]

with quantized energy and integration constant

\[
E = -Q^2 (2(n + km) + 1 + ka + kb)^{-2} \quad \quad A = \frac{1}{4} k^2 (2m + a + b)^2
\]

renormalization of constants \(\alpha = a(a - 1), \beta = b(b - 1)\) and gauges

\[
G_1 = r^{\sqrt{A}} e^{2r\sqrt{-E}} \quad \quad G_2 = \cos\left(\frac{k}{2} \phi\right)^a \sin\left(\frac{k}{2} \phi\right)^b
\]

For a given rational \(k = c/d\) the energy levels are indexed by an integer \(N = dn + cm\) and their degeneracy is \(D = [dN/c] + 1\). This coincides with the degeneracy of an anisotropic oscillator with frequency ratio \(c/d\).
Integrability

The deformed Coulomb potential admits separation of variables in polar coordinates and hence has a second-order integral of motion given by

\begin{align*}
L_1 &= p^2_\phi + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2} \phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2} \phi\right)} \\
L_1 &= -\partial^2_\phi + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2} \phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2} \phi\right)}
\end{align*}

The deep connection between these systems is a direct result of the fact that they are related via the Coupling Constant Metamorphosis.
Integrability

The deformed Coulomb potential admits separation of variables in polar coordinates and hence has a second-order integral of motion given by

\[ L_1 = p_\phi^2 + \frac{\alpha k^2}{4r^2 \cos^2 \left( \frac{k}{2} \phi \right)} + \frac{\beta k^2}{4r^2 \sin^2 \left( \frac{k}{2} \phi \right)} \]

\[ L_1 = -\partial_\phi^2 + \frac{\alpha k^2}{4r^2 \cos^2 \left( \frac{k}{2} \phi \right)} + \frac{\beta k^2}{4r^2 \sin^2 \left( \frac{k}{2} \phi \right)} \]

The deep connection between these systems is a direct result of the fact that they are related via the Coupling Constant Metamorphosis.
Classical deformation

We can apply CCM to potentials which separate in polar coordinates with radial part containing an oscillator term and relate it to one with a Coulomb term.

Theorem

Given a classical Hamiltonian of the form

\[ \mathcal{H}(\rho, \theta) = p_\rho^2 + \frac{1}{\rho^2} p_\theta^2 - \tilde{E} \rho^2 + f_1(\rho) + \frac{1}{\rho^2} f_2(\theta) \]

where \( f_1(r) \) and \( f_2(\theta) \) are independent of \( \tilde{E} \). We have a transformed system, again on the Euclidean plane

\[ \tilde{\mathcal{H}}(r, \phi) = p_r^2 + \frac{1}{r^2} p_\phi^2 - \frac{E}{2r} + \frac{1}{2r} f_1(\sqrt{2r}) + \frac{1}{4r^2} f_2\left(\frac{\phi}{2}\right) \]

and the integrals of motion for \( \mathcal{H} \) will be mapped to integrals of motion for \( \tilde{\mathcal{H}} \).
Classical Trajectories

Furthermore, the classical trajectories will be related via coordinate change and a reparameterization of the time.

**Theorem**

If $\mathcal{H}(\rho, \theta)$ as given above has trajectories $(\rho(t), \theta(t))$ which satisfy

$$\delta_1 = F_1(\rho) - t, \quad \delta_2 = F_2(\rho, \theta)$$

then trajectories for $\tilde{\mathcal{H}}(r, \phi)$ will satisfy

$$\delta_1 = \frac{d}{d\tilde{E}} \int F_1(\sqrt{2r})dE - t, \quad \delta_2 = F_2(\sqrt{2r}, \frac{\phi}{2}).$$
Quantum Deformation

We have analogous results for the quantum system

**Theorem**

\[
H(\rho, \theta) = -\frac{1}{\rho} \partial_\rho (\rho \partial_\rho) - \frac{1}{\rho^2} \partial_\theta^2 - \tilde{E} \rho^2 + f_1(\rho) + \frac{1}{\rho^2} f_2(\theta)
\]

where \(f_1(\rho)\) and \(f_2(\theta)\) are independent of \(\tilde{E}\). We have a transformed system, again on the Euclidean plane

\[
\tilde{H}(r, \phi) = -\frac{1}{r} \partial_r (r \partial_r) - \frac{1}{r^2} \partial_\phi^2 - \frac{E}{2r} + \frac{1}{2r} f_1(\sqrt{2r}) + \frac{1}{4r^2} f_2\left(\frac{\phi}{2}\right)
\]

and any integrals of motion, of the form \(L = \sum_{j=0}^{\left[\frac{N}{2}\right]} K_{N-2j} \tilde{E}^j\), for \(H\) will be mapped to integrals of motion for \(\tilde{H}\).
Quantum Wave functions

The relation between the wave functions is even more direct, given by

**Theorem**

If \( \psi(\rho, \theta) \) is a solution to \( H(\rho, \theta)\psi(\rho, \theta) = E\psi(\rho, \theta) \) then \( \psi(\sqrt{2r}, \phi) \) will be a solution to \( \tilde{H}(r, \phi)\psi(\sqrt{2r}, \phi) = \tilde{E}\psi(\sqrt{2r}, \phi) \).

So we see that two systems related via CCM are intimately related and the analysis of the systems are essential identical and any potential with a oscillator terms is CCM equivalent to one with a Coulomb term, both on the Euclidean plane.
Quantum Wave functions

The relation between the wave functions is even more direct, given by

Theorem

If $\psi(\rho, \theta)$ is a solution to $H(\rho, \theta)\psi(\rho, \theta) = E\psi(\rho, \theta)$ then $\psi(\sqrt{2r}, \frac{\phi}{2})$ will be a solution to $\tilde{H}(r, \phi)\psi(\sqrt{2r}, \frac{\phi}{2}) = \tilde{E}\psi(\sqrt{2r}, \frac{\phi}{2})$.

So we see that two systems related via CCM are intimately related and the analysis of the systems are essential identical and any potential with a oscillator terms is CCM equivalent to one with a Coulomb term, both on the Euclidean plane.
The Deformed Coulomb and the TTW systems

If we take, as in the previous theorems

\[ \tilde{E} = -\omega^2, \quad E \equiv Q \]

\[ f_1(\rho) = 0 \quad f_2(\theta) = \alpha \frac{k^2}{\rho^2 \cos^2(k\theta)} + \beta \frac{k^2}{\rho^2 \sin^2(k\theta)} \]

then the original Hamiltonians \( \mathcal{H}, H \) corresponds to the TTW system, \( \mathcal{H}_{k^{TTW}}, H_{k^{TTW}} \) and the transformed system corresponds to the deformed Coulomb system

\[ \mathcal{H}^{TTW} = \mathcal{H}^{DC} \]

\[ \mathcal{H}^{DC} = p_r^2 + \frac{1}{r} p_\theta^2 - \frac{Q}{r} + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2}\phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2}\phi\right)}. \]

with similar results for the quantum Hamiltonian.
Comparison

If we compare these to the bounded trajectories of the TTW system, we obtain

**Deformed Coulomb**

\[
0 = \frac{-2Er - Q}{\sqrt{Q^2 + 4AE}} + \sin \left( \frac{4(-E)^{3/2}}{Q}(t + \delta_1) + \frac{2\sqrt{-E}}{a} \sqrt{Er^2 + Qr - A} \right)
\]

\[
\delta_2 = \frac{1}{2\sqrt{A}} \arcsin(R) + \frac{1}{2k\sqrt{A}} \arcsin(U_k).
\]

\[
R = \frac{2A - Qr}{r\sqrt{Q^2 + 4AE}}, \quad U_k = \frac{2A \sin^2\left(\frac{k}{2}\phi\right) - A + \frac{k^2}{4}(\beta - \alpha)}{\sqrt{(A - \frac{k^2}{4}(\beta + \alpha))^2 - \frac{\alpha\beta k^4}{4}}}.
\]

**TTW**

\[
\rho^2 = \frac{1}{2\omega^2} \left( E + \sqrt{(E^2 - 4\omega^2 A)} \sin[4\omega(t + \delta_1)] \right)
\]

\[
k \arcsin R + \arcsin U_k = -4k\sqrt{A}\delta_2
\]

\[
R = \frac{-2A + E\rho^2}{\rho^2 \sqrt{E^2 - 4\omega^2 A}}, \quad U_k = \frac{-2A \sin^2\theta - A - (\alpha - \beta)k^2}{\sqrt{(A - (\alpha + \beta)k^2)^2 - 4\alpha\beta k^2}}.
\]
Deformed Coulomb

\[ \Psi = G_1 G_2 L_n^{k(2m+a+b)} \left( 2r \sqrt{-E} \right) P_m^{a-\frac{1}{2}, b-\frac{1}{2}} (- \cos(k\phi)) \]

\[ \begin{align*}
G_1 &= r^{km} r^{k(a+b)/2} e^{2r \sqrt{-E}} \\
G_2 &= \cos \left( \frac{k}{2} \phi \right)^a \sin \left( \frac{k}{2} \phi \right)^b
\end{align*} \]

TTW

\[ \Phi = \rho^{2nk} L_N^{(k(2n+\alpha+\beta))} (\omega \rho^2) P_n^{(a-1/2, b-1/2)} (2 \sin^2 k\theta - 1), \]

\[ G = \rho^{(a+b)k} \cos^a k\theta \sin^b k\theta e^{-\frac{\omega r^2}{2}} \]
Integrability

The deformed Coulomb potential admits separation of variables in polar coordinates and hence has a second-order integral of motion given by

\[ L_1 = p_{\phi}^2 + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2} \phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2} \phi\right)} \]

\[ L_1 = -\partial_{\phi}^2 + \frac{\alpha k^2}{4r^2 \cos^2\left(\frac{k}{2} \phi\right)} + \frac{\beta k^2}{4r^2 \sin^2\left(\frac{k}{2} \phi\right)} \]

which we can compare with the second-order integrals for the TTW system

\[ L_1^{(TTW)} = p_\theta^2 + \frac{\alpha k^2}{r^2 \cos^2(k\theta)} + \frac{\beta k^2}{r^2 \sin^2(k\theta)} \]

\[ L_1^{(TTW)} = -\partial_\theta^2 + \frac{\alpha k^2}{r^2 \cos^2(k\theta)} + \frac{\beta k^2}{\sin^2(k\theta)} \]
Integrability

The deformed Coulomb potential admits separation of variables in polar coordinates and hence has a second-order integral of motion given by

\[ L_1 = p^2_\phi + \frac{\alpha k^2}{4r^2 \cos^2(\frac{k}{2} \phi)} + \frac{\beta k^2}{4r^2 \sin^2(\frac{k}{2} \phi)} \]

\[ L_1 = -\partial^2_\phi + \frac{\alpha k^2}{4r^2 \cos^2(\frac{k}{2} \phi)} + \frac{\beta k^2}{4r^2 \sin^2(\frac{k}{2} \phi)} \]

which we can compare with the second-order integrals for the TTW system

\[ L_1^{(TTW)} = p^2_\theta + \frac{\alpha k^2}{r^2 \cos^2(k\theta)} + \frac{\beta k^2}{r^2 \sin^2(k\theta)} \]

\[ L_1^{(TTW)} = -\partial^2_\theta + \frac{\alpha k^2}{r^2 \cos^2(k\theta)} + \frac{\beta k^2}{\sin^2(k\theta)} \]
Superintegrability of the Deformed Coulomb system

Kalnins, Miller and Pogosyan proved the superintegrability of the classical TTW system for rational $k$ in a recent paper (to appear, arXiv:0912.2278v1) demonstrating higher order integrals which are polynomial both in the momentum and in the coupling constant.

Kalnins, Kress and Miller proved the superintegrability of the quantum TTW system for rational $k$ in another paper (2010 J. Phys. A: Math. Theor. 43 265205) though it is still an open question whether the integrals have the appropriate form for CCM, for all rational $k$, though no counter examples exist.
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The coupling constant metamorphosis is a important tool for classifying superintegrable systems as well as creating new systems, possibly on different ambient manifolds.

The representation theory of the equivalence classes give information about the physical systems and are interesting objects in their own right, especially regarding special functions.

The existence of families of superintegrable systems which are deformations of the simple harmonic oscillator and the Coulomb potential, the only two potential which are radially symmetric and whose bounded trajectories must be closed (J. Bertrand. C. R. Acad. Sci., 77:847-853, 1873).

Thank you for listening.