Quasi-exact solvability: before and after.

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Symmetries of Differential Equations
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Talk overview

- Quasi-exact solvability
- Lie algebraic operators
- A bit of history
- Exceptional subspaces
- Example: multiple algebraic sectors
- Exceptional orthogonal polynomials.
- Concluding remarks: recent developments, future outlook.
Exactly and Quasi-exactly solvable potentials

**ES potentials**: all eigenvalues algebraic, invariant polynomial flag

- Quantum harmonic oscillator: \(-\psi''(x) + x^2\psi(x) = \lambda\psi(x)\)
- Hermite polynomials: \(y''(z) - 2zy'(z) + ny(z) = 0\)
- Change of variables: \(z = x, \psi(x) = e^{-\frac{z^2}{2}}y(z), \lambda = n + 1\)
- **Observation**: Hermite operator \(D_{zz} - 2zD_z\) preserves the standard polynomial flag 1, \(z, z^2\ldots\)

**QES potentials**: finite “algebraic” eigenvalues: invariant subspaces

- sextic potential: \(U(x) = x^6 + 4cx^4 + (4c^2 - 4n - 3)x^2\)
- change of var: \(x = z^2, \psi(x) = e^{-z^2/4-cz}y(z)\)
  \[
  T[y] := 4zy'' + 2(1 - 4cz)y' + \lambda y + 4(nzy - z^2y') = 0
  \]
- \(T\)-invariant subspace: \(P_n = \langle 1, z, \ldots, z^n \rangle\)
  \[
  T[z^n] = (\lambda - 8cn)z^n + \text{lower deg. terms}
  \]
  \(M = [T|P_n]\), tridiagonal matrix
- Algebraic eigenvalues: \(\chi_M(\lambda) = 0\)
Lie algebraic operators

- SL$_2$ action: $\hat{z} = \frac{az + b}{cz + d}$, $z = \frac{d\hat{z} - b}{a - c\hat{z}}$, $ad - bc = 1$
- Polynomial representation: $\hat{p}(\hat{z}) = (a - c\hat{z})^n p(z)$, $p(z) \in \mathcal{P}_n$
- Infinitesimal sl$_2$ counterpart: 1st order operators

\[ T_0 = Dz, \quad T_1 = 2zDz - n, \quad T_2 = z^2Dz - nz \]

\[ [T_0, T_1] = 2T_0, \quad [T_0, T_2] = T_1, \quad [T_1, T_2] = 2T_2 \]

- Lie algebraic operator: $T \in \mathcal{U}(sl_2)$. Preserves $\mathcal{P}_n$ by construction.
- **Burnside's Theorem**: every differential operator that preserves $\mathcal{P}_n$ belongs to $\mathcal{U}(sl_2)$.

| Lie algebraic operator + change of variables + physical constraints | $\Rightarrow$ QES operators. |
A bit of history

By the middle of the 80’s, Shifman, Turbiner, and Ushveridze, had introduced the basic definition of quasi-exactly solvability. My own interest in the subject began with a provocative lecture given by Raphael Levine, at the Institute for Mathematics and its Applications, Minnesota, in 1987. The classification problems raised by Levine seemed ideally suited to the equivalence methods that Niky Kamran and I were developing at that time. A couple of years later, we had the good fortune to also enlist Artemio González-López in this enterprise, and the resulting collaboration proved to be extraordinarily fertile.

-Peter Olver, “A QES Travel Guide”, 1997

- Multi-dimensional operators, multi-particle systems.
- Multi-spectral problems and super-integrable/multi-separable systems.
- QES difference equations.
- Multi-component systems, spin-chains.

Peter’s question: are all (scalar) QES potentials Lie algebraic?
The direct approach to QES.

Rational operators with polynomial eigenfunctions.

- **Example:** \( T[y] = y'' - \frac{2}{z}((z + 1)y' - y) \) preserves \( \mathcal{E}_n = \langle z + 1, z^2, \ldots, z^n \rangle \)

- **Exceptional operator:** a second-order operator that leaves invariant a finite-dimensional, primitive (no common factors) polynomial subspace \( U \subset \mathcal{P}_n \), but does not preserve \( \mathcal{P}_n \)

- **Exceptional subspace:** maximal invariant polynomial subspace of an exceptional operator.

- **Theorem [GU,K,M]:** every exceptional codimension 1-subspace is \( SL_2 \)-equivalent to \( \mathcal{E}_n \).

- **Application: codimension 1 QES.** Classify all 2nd order operators that preserve \( \mathcal{E}_n \) and write down all physically interesting potentials. By construction, such potentials will not be Lie-algebraic.
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The exceptional marriage of even and odd algebraic sectors.

- Schrodinger equation: $-\psi''(x) + U(x)\psi(x) = \lambda\psi(x)$
- QES potential: $U(x) = 2A^2 \cosh(2x) + 4An\cosh(x) - \frac{1}{2} \sech^2(x/2)$
- SL$_2$ operators: $z = -\sinh^2(x/2)$, 
  
  $y_i(z) = e^{2A\cosh(x)} \cosh^{1-2i}(x/2) \sinh^i(x)\psi(x), \ i = 0, 1$
  
  $T_0[y] = z(z-1)y''' + (8Az(1-z) - \frac{1}{2})y' + 8Anzy$
  
  $T_1[y] = z(z-1)y''' + (8Az(1-z) - 4z - \frac{3}{2})y' + 8A(n-2)zy$
- Exceptional operator: $w = e^x$, 
  
  $Y(w) = e^{2A\cosh(x) - nx} \sech(x/2)\psi(x)$
  
  $w^2 Y''' + 2(Aw^2 - 2w + (1 - A))Y' - 4AnwY - \frac{2Y' + 2nY}{1 + w} + \lambda Y = 0,$
- Algebraic change of variables:
  
  $z = -\frac{(w-1)^2}{4w}, \ y_0(z) = w^{-n}Y(w), \ y_1(z) = \frac{w^{2-n}}{(w-1)(w+1)^3}aY(w)$

SL$_2$ Invariant subspaces:

$y_0 \in \langle 1, z, z^2, \ldots, z^n \rangle$, $y_1 \in \langle 1, z, z^2, \ldots, z^{n-2} \rangle$,

Exceptional invariant subspace:

$Y \in \langle w + 1 - \frac{1}{n}, (w + 1)^2, (w + 1)^3, \ldots, (w + 1)^{2n} \rangle$
Hidden algebraic sector cont.

\[ U[x] = 2A^2 \cosh(2x) + 4A n \cosh(x) - \frac{1}{2} \text{Sech}(x/2)^2 \]

\[ A = -1, \quad n = 2 \]
Hidden algebraic sector cont.

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\[ \begin{array}{c|c}
\text{sl(2) algebraization} & \text{Exceptional algebraization} \\
\hline \\
\end{array} \]
Exceptional orthogonal polynomials

- Classical ES potentials correspond to classical orthogonal polynomials.
- An exactly solvable $X_1$ potential arises when an operator preserves the unique $X_1$ flag: $z + 1, z^2, z^3, \ldots$
- **Definition:** an $X_1$ orthogonal polynomial family is a polynomial sequence starting with *degree 1* that is complete and orthogonal relative to a measure associated to a Sturm-Liouville problem.
- **Theorem [GU,K,M]** The $X_1$ Laguerre and $X_1$ Jacobi families are the unique codimension-1 orthogonal polynomials.
- **$X_1$ Laguerre.**
  - Measure: $\frac{x^k e^{-x}}{(x + k)^2} dx, \quad k > 0, \quad x > 0$
  - ODE: $xy'' - \left( \frac{x-k}{x+k} \right) ((x+k+1)y' - y) + (n-1)y = 0, \quad n > 1$
  - Flag: $x + k + 1, (x + k)^2, (x + k)^3, \ldots, (x + k)^n, \ldots$
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Recent developments. Future outlook.

- Exceptional orthogonal polynomials give rise to new examples of shape-invariant potentials.
  - Examples of arbitrarily high codimension are known.
  - All known examples arise as the application of the Darboux-Crum iterated transformation to classical ES potentials.
- Conjecture [GU,K,M]: all exceptional orthogonal polynomials (and by extension ES potentials) are generated by the iterated application of the Darboux-Crum transformation to the classical families.
  - Conjecture has been verified for codimensions 1 and 2.
  - Applications to QES await!

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Happy birthday Peter!
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