A Numerical Study of the Focusing Davey-Stewartson II Equation

Christian Klein
Benson Muite
Kristelle Roidot

University of Michigan
muite@umich.edu
www.math.lsa.umich.edu/~muite

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The Davey-Stewartson Equation
- Original Derivation
- Properties of the Solutions

Previous Analytical Work
- Well Posedness
- Exact Solutions

Previous Numerical Studies
- The Davey-Stewartson Equation
- The 1D Quintic Nonlinear Schrödinger Equation
- Summary

Current Work
- Goal
- Description of Numerical Methods
- Numerical Results
- Conclusion
The Davey Stewartson System: 1

\[ iu_t + u_{xx} - \alpha u_{yy} + 2\rho \left( \phi + |u|^2 \right) u = 0 \]
\[ \phi_{xx} + \beta \phi_{yy} + 2|u|_{xx}^2 = 0 \]

\( \alpha, \beta, \rho = \pm 1 \)

- Focusing \( \rho = -1 \), Defocusing \( \rho = 1 \)
- Integrable when \( \alpha = \beta \)
- We consider the case \( \rho = -1 \) and \( \alpha = \beta = 1 \) which is known as focusing DS II equation
- The other integrable cases are
  1. \( \rho = 1 \) and \( \alpha = \beta = 1 \) defocusing DS II equation
  2. \( \rho = 1 \) and \( \alpha = \beta = -1 \) defocusing DS I equation
  3. \( \rho = -1 \) and \( \alpha = \beta = -1 \) focusing DS I equation
First derived by Davey and Stewartson in 1974

Formal derivation based on multiple scales expansion for the free surface of an inviscid irrotational three dimensional fluid

Solution only valid for short times

Solution is valid in the shallow water limit when surface tension and gravity are both important.

For an introduction see Sulem and Sulem
The Nonlinear Schrödinger Equation
Properties of the Davey-Stewartson II equation

- Integrable – can be solved by an inverse scattering transform
- Important conserved quantities for regular enough solutions on $\mathbb{R}^2$ and $\mathbb{T}^2$ are
  1. Mass $\|u\|_2^2$
  2. Momentum $\iint i (u^* \nabla u - u \nabla u^*) \, dx \, dy$
  3. Energy
     \[
     \frac{1}{2} \iint \left[ |\partial_x u|^2 - |\partial_y u|^2 - \rho \left( |u|^4 - \frac{1}{2} \phi^2 + (\partial_x^{-1} \partial_y \phi)^2 \right) \right] \, dx \, dy
     \]

Sung (1994, 1995) Global existence, uniqueness and decay for $u_0 \in L^p$, $1 \leq p < 2$, $\hat{u}_0 \in L^1 \cap L^\infty$ and

$$
\| \hat{u}_0 \|_{L^1} \| \hat{u}_0 \|_{L^\infty} < \frac{\pi^3}{2} \left( \frac{\sqrt{5} - 1}{2} \right)^2 \approx 5.92,
$$

where $\hat{u}_0$ is the Fourier transform of the initial data $u_0$. Proofs use inverse scattering transform.
Relevant Exact Solutions to the Focusing Davey-Stewartson II Equation

- The single lump solution (Arkadiev, Pogrebkov and Polvianov, 1989)

\[ u = 2 \exp(-2i(\zeta x - \eta y + 2(\zeta^2 - \eta^2)t)) \]
\[ \frac{|x + 4\zeta t + i(y + 4\eta t) + 1|^2 + 1}{|x + 4\zeta t + i(y + 4\eta t) + 1|^2 + 1} \]

- Ozawa’s blow up solution (Ozawa, 1992)

\[ u = \exp \left( i \frac{x^2 - y^2}{1 - 4t} \right) \frac{(1 - 4t)}{(1 - 4t)^2 + x^2 + y^2} \]
Open Analytical Questions related to the Focusing Davey-Stewartson II Equation

- What is the generic behavior for solutions not satisfying the Sung condition?
- What determines whether a solution which does not satisfy the Sung condition blows up, disperses or has soliton behavior?
- Is there a “good” means of extending solutions past the blow up time if they do blow up?
Previous Numerical Studies of the Focusing Davey-Stewartson II Equation

- **White and Weideman (1994)**
  1. Showed a Fourier pseudo spectral splitting scheme could be used to simulate the Davey-Stewartson II equation

- **Besse, Mauser and Stimming (2004)**
  1. Used a parallel code to examine numerical solutions to the focusing Davey-Stewartson II equation
  2. Upto $4096^2$ grid points, high spatial and temporal resolution required to get correct results
  3. Commented that a splitting scheme could step over the blow up, but no conclusive results given

- **McConnell, Fokas and Pelloni (2005)**
  1. Used a Matlab code to examine numerical solutions to the focusing Davey-Stewartson II equation
  2. Spatial resolution was not high enough to obtain conclusive results
  3. Suggested that a splitting scheme could step over the blow up
Previous Relevant Numerical Studies of the Focusing 1D Quintic Nonlinear Schrödinger Equation

- Stinis (2012)

\[ i\psi_t + \psi_{xx} + |\psi|^4\psi = 0 \]

1. Existence of non unique global solutions in \( L^\infty L^2 \)
2. Possible blow up of solutions in \( L^\infty H^1 \)
3. Computes blow up solutions using a pseudo spectral method and chooses to enforce energy conservation to pick a solution after the blow up time
Numerical methods can be used to suggest that solutions to semilinear dispersive equations blow up.

Care is required in their use, in particular sufficient spatial and temporal resolution is required.

The choice of numerical method is important when equations do not have unique solutions.

If a numerical scheme steps over blowup, how does one determine that blowup has occurred?

How does one distinguish between blow up and insufficient resolution?
Write a parallel Fourier pseudospectral code for the Davey Stewartson equations

Verify that the program works correctly

Find a method which allows one to distinguish between numerical simulations of blow up and under resolved simulations

Perform high resolution simulations to examine which initial data are likely to lead to blow up
The Split Step Fourier Method 1: The Quintic Nonlinear Schrödinger Equation

\[ iu_t + |u|^4 u + u_{xx} = 0 \]

- **Strang or 2\textsuperscript{nd} order splitting. A 4\textsuperscript{th} order method is similar.**
  1. First solve
     \[ iu_t + u_{xx} = 0 \]
     exactly in Fourier space to get \( u(\delta t/2, \cdot) \).
  2. Then solve
     \[ iu_t + |u|^4 u = 0 \]
     exactly in real space using \( u(\delta t/2, \cdot) \) as initial data and the fact that \( |u|^2 \) is a conserved quantity.
  3. Finally solve
     \[ iu_t + u_{xx} = 0 \]
     exactly in Fourier space to get \( u(\delta t, \cdot) \) using the initial data produce at the second stage.
  4. Repeat as many times as necessary to get to the final time of interest
Although this method does work, convergence has only been shown for quite smooth solutions.

Relevant works include Lubich and Gauckler (2010), Faou (2012).

DEMO

Blow up time in agreement with Stinis (2012) for the same initial data, but using a different numerical method.
By tracing the energy we can determine when blow up/under resolution occurs. Experimentally, for this example blowup occurs quickly, whereas under resolution occurs slowly. The numerical method continues to produce what seems to be a solution after the blow up time.
The Split Step Fourier Method 4: The focusing Davey-Stewartson II Equation

- Demo for lump solution
- Demo for Ozawa’s solution
Mass and energy are conserved for smooth solutions.

Energy again shows a change when there is blow up, but mass is conserved.

Higher spatial and temporal resolution required to better determine if blow up: use parallel resources.
Parallel scalability on Kraken for $49152^2$ grid points using a 4th order split step method.
The Split Step Fourier Method 7: The focusing Davey-Stewartson II Equation

- Ozawa
- Blowup
Summary of findings for focusing Davey-Stewartson II system

- Perturbations of the Ozawa and lump solutions with added energy blowup
- Perturbations of the Ozawa and lump solutions with reduced energy disperse
- More analytical and numerical work needed to figure out exact conditions that lead to blow up or to dispersion
- In two dimensions, resolution is at present limited by numerical precision
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