Singularity formation in critical parabolic problems I

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Let $n \geq 3$. The energy critical heat equation in $\mathbb{R}^n$ is the Cauchy parabolic problem

$$\begin{cases}
  u_t = \Delta u + |u|^\frac{4}{n-2} u & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\
  u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^n.
\end{cases}$$

(P0)

The energy of problem (P0)

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{n-2}{2n} \int_{\mathbb{R}^n} |u|^\frac{2n}{n-2}$$

is a Lyapunov functional for finite energy classical solutions

$$\frac{d}{dt} E(u(\cdot, t)) = - \int_{\mathbb{R}^n} |u_t|^2 \leq 0.$$ 

Classical theory yields well-posedness of (P0) for short times.
We consider **positive finite-energy solutions** which are globally defined time. The presence of the Lyapunov functional implies that limits along sequences $t = t_n \to +\infty$ can only be steady states:

$$
\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n.
$$

All **positive** solutions are the *Aubin-Talenti bubbles*

$$
U_{\mu, \xi}(x) = \mu^{-\frac{n-2}{2}} U \left( \frac{x - \xi}{\mu} \right),
$$

where $\mu > 0$, $\xi \in \mathbb{R}^n$ and

$$
U(x) = \alpha_n \left( \frac{1}{1 + |x|^2} \right)^{\frac{n-2}{2}}, \quad \alpha_n = (n(n-2))^{\frac{n-2}{4}}.
$$

These are precisely the extremals of Sobolev’s embedding.
The criticality of Problem \((P0)\) refers to the presence of this continuum of steady states which become singular as \(\mu \to 0\) and are energy invariant: \(E(U_{\mu,\xi}) = E(U)\) for all \(\xi \in \mathbb{R}^n, \mu > 0\).

\[ U_{\mu,\xi}(x) = \alpha_n \left( \frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}}, \quad \mu \downarrow 0 \]
A bubbling blow-up solution of \((P0)\) is one which looks like

\[ u(x, t) \approx \sum_{j=1}^{k} U_{\mu_j(t), \xi_j(t)}(x), \quad \mu_j(t) \to 0. \]
The exponent $p = p_S := \frac{n+2}{n-2}$ is special in the more general equation

$$u_t = \Delta u + |u|^{p-1} u = 0 \quad \text{in } \mathbb{R}^n$$

for which the following energy is a Lyapunov functional.

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1}$$

- No positive stationary solution exists if $p < p_S$.
- All positive stationary solutions for $p = p_S$ are the bubbles $U_{\mu, \xi}$.
- For $p > p_S$ there are positive stationary solutions with $u(x) \sim |x|^{-2}$. They do not have finite energy.
Bubbling phenomena of the kind described is present in many important time-dependent and stationary settings usually carrying deep meaning in the global structure of their solutions. Notable examples extensively studied include:

- The harmonic map problem and the harmonic map flow.
- The Yamabe problem and the Yamabe flow.
- The Keller-Segel chemotaxis system.
- Blow-up in Energy-critical wave equations, Schrodinger maps and other dispersive settings.

Key in understanding these phenomena is the construction of bubbling solutions. This is usually not an easy matter. The energy critical heat equation is a model that reflects many of the difficulties arising.
First we consider the version of problem \((P0)\) in a bounded, smooth domain \(\Omega \subset \mathbb{R}^n, \ n \geq 3\). We construct global solutions of the critical semilinear heat equation

\[
\begin{aligned}
&u_t = \Delta u + |u|^\frac{4}{n-2} u \\ &u = 0 \\ &u = u_0 > 0
\end{aligned}
\text{in}\ \Omega \times (0, \infty),
\text{on}\ \partial\Omega \times (0, \infty),
\text{in}\ \Omega \times \{0\}. \quad (P)
\]

with infinite-time bubbling blow-up.

What is the behavior of a globally defined solution (if any)?
• It follows from Struwe (1984) (see also Suzuki 2003) that a global solution of \((P)\) \(u(x, t)\) satisfies along sequences \(t_n \to +\infty\) a bubble resolution of the form

\[
u(x, t_n) = u_\infty(x) + \sum_{i=1}^{k} U_{\mu_i^n, \xi_i^n}(x) + o(1),
\]

for some \(k \geq 0\), a stationary solution \(u_\infty\) and \(\xi_i^n \in \Omega, \mu_i^n \to 0\).

• Necessarily \(u_\infty = 0\) if \(\Omega\) is star-shaped thanks to Pohozaev’s identity and

\[
\lim_{t \to +\infty} E(u(\cdot, t)) = kS_n, \quad S_n = E(U).
\]

Are there such solutions with \(k \geq 1\)?
• The only known example: Galaktionov and King (2002) \( \Omega = B(0,1) \). There is a radial solution \( u(|x|, t) \) with

\[
\lim_{t \to +\infty} E(u(\cdot, t)) = S_n, \quad u(|x|, t) = U_{\mu(t),0}(x) + o(1)
\]

where for \( n \geq 5, \mu(t) = ct^{-\frac{1}{n-4}} \to 0 \). The proof uses radial symmetry in essential way.

• For a general domain and any \( k \geq 1 \), the natural question is: Are there globally defined solutions \( u(x, t) \) to \( (P) \) with

\[
\lim_{t \to \infty} E(u(\cdot, t)) = kS_n ?
\]

We prove that the answer is yes.
We consider the Green function for problem

$$-\Delta_x G(x, y) = c_n \delta(x - y) \quad \text{in } \Omega, \quad G(\cdot, y) = 0 \quad \text{on } \partial\Omega,$$

where $\delta(x)$ is the Dirac mass at the origin and $c_n$ is the number such that

$$-\Delta_x \Gamma(x) = c_n \delta(x), \quad \Gamma(x) = \frac{\alpha_n}{|x|^{n-2}},$$

with $\alpha_n$ the number in the bubble’s definition. We let $H(x, y)$ be the regular part of $G(x, y)$ namely the solution of the problem

$$-\Delta_x H(x, y) = 0 \quad \text{in } \Omega, \quad H(\cdot, y) = \Gamma(\cdot - y) \quad \text{in } \partial\Omega.$$
The diagonal $H(x, x)$ is called the Robin function of $\Omega$. It is well known that it satisfies

$$H(x, x) \to +\infty \text{ as } \text{dist}(x, \partial\Omega) \to 0.$$ 

Let $q = (q_1, \ldots, q_k)$ be an array of $k$ distinct points in $\Omega$, and define the $k \times k$ matrix

$$G(q) = \begin{bmatrix}
H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\
-G(q_1, q_2) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\
\vdots & \vdots & \ddots & \vdots \\
-G(q_1, q_k) & \cdots & -G(q_{k-1}, q_k) & H(q_k, q_k)
\end{bmatrix}$$
Theorem (Cortázar, del Pino, Musso, 2016)

Let $n \geq 5$, $k \geq 1$.

Let $q_1, \ldots, q_k$ be points such that the matrix $G(q)$ is positive definite. There exists a global solution $u_q(x, t)$ of problem $(P)$ of the form

$$u_q(x, t) = \sum_{j=1}^{k} U_{\mu_j, \xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x, q_j) + o(\mu_j^{\frac{n-2}{2}})$$

as $t \to +\infty$, where for certain numbers $\beta_j > 0$

$$\mu_j(t) = \beta_j t^{-\frac{1}{n-4}} (1 + o(1)), \quad \xi_j(t) - q_j = O(t^{-\frac{2}{n-4}}).$$
Our construction of the solution \( u_q(x, t) \) yields the \textbf{codimension} \( k \)-\textbf{stability} of its bubbling phenomenon in the following sense.

\textbf{Corollary}

There exists a codimension \( k \) manifold in \( C^1(\bar{\Omega}) \) that contains \( u_q(x, 0) \) such that if \( u(x, 0) \) lies in that manifold and it is sufficiently close to \( u_q(x, 0) \), then the solution \( u(x, t) \) of \( (P) \) has exactly \( k \) bubbling points \( \tilde{q}_j, j = 1, \ldots, k \) which lie close to the \( q_j \).
Positive definiteness of $G(q)$ trivially holds if $k = 1$. For $k = 2$ this condition holds if and only if

$$H(q_1, q_1)H(q_2, q_2) - G(q_1, q_2)^2 > 0,$$

in particular it does not hold if both points $q_1$ and $q_2$ are too close to a given point in $\Omega$. Given $k > 1$, using that

$$H(x, x) \rightarrow +\infty \quad \text{as} \quad \text{dist}(x, \partial \Omega) \rightarrow 0.$$

we can always find $k$ points where $G(q)$ is positive definite.
Construction of an approximate solution

Given \( k \) points \( q_1, \ldots, q_k \in \mathbb{R}^n \), we want to find a solution \( u(x, t) \) of equation (P) with

\[
u(x, t) \approx \sum_{j=1}^{k} U_{\mu_j(t), \xi_j(t)}(x)
\]

where \( \xi_j(t) \to q_j \) and \( \mu_j(t) \to 0 \) as \( t \to \infty \) for each \( j = 1, \ldots, k \).

We assume: For a function \( \mu_0(t) \to 0 \) and constants \( b_1, \ldots, b_k \)

\[
\mu_j(t) = b_j \mu_0(t) + O(\mu_0^2) \quad \text{as } t \to \infty,
\]

\[
\xi_j(t) = q_j + O(\mu_0^2) \quad \text{as } t \to \infty.
\]
If a solution to \((P)\) satisfies \(u(x, t) \approx \sum_{j=1}^{k} U_{\mu_j, \xi_j}(x)\) then

\[
    u_t \approx \Delta u + \sum_{j=1}^{k} U_{\mu_j, q}(x)^p
\]

Besides, we see that

\[
    \int_{\Omega} U_{\mu_j, q}(x)^p \, dx \approx \mu_j^{\frac{n-2}{2}} a_n, \quad a_n := \int_{\mathbb{R}^n} U(y)^p \, dy,
\]

and hence away from the points \(q_j\)

\[
    u_t \approx \Delta u + c_n \mu_0^{\frac{n-2}{2}} \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0, \infty).
\]

where \(\delta_q\) is the Dirac mass at the point \(q\).
Letting $u = \mu_0^{\frac{n-2}{2}} v(x, t)$ we get

$$v_t \approx \Delta v - \frac{n-2}{2} \mu_0^{-1} \dot{\mu}_0 v + c_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0, \infty).$$

We assume that $\mu_0^{-1} \dot{\mu}_0 \to 0$, so that

$$v_t \approx \Delta v + a_n \sum_{j=1}^k b_j^{\frac{n-2}{2}} \delta_{q_j} \quad \text{in } \Omega \times (0, \infty),$$

$$v = 0 \quad \text{on } \partial \Omega \times (0, \infty).$$
So that away from the $q_j$ we should have

$$v(x, t) \approx a_n \sum_{j=1}^{k} b_j^{\frac{n-2}{2}} G(x, q_j),$$

$$u(x, t) \approx \sum_{j=1}^{k} \frac{\alpha_n \mu_j^{\frac{n-2}{2}}}{|x - q_j|^{n-2}} - \mu_j^{\frac{n-2}{2}} H(x, q_j).$$
Observing that for $x$ away from the point $q_j$, we precisely have

$$U_{\mu_j, \xi_j}(x) \approx \frac{\alpha n \mu_j^{\frac{n-2}{2}}}{|x - q_j|^{n-2}}$$

we see that a better global approximation to a solution $u(x, t)$ to our problem is given by the corrected $k$-bubble

$$u_{\xi, \mu}(x, t) := \sum_{j=1}^{k} u_j(x, t), \quad u_j(x, t) := U_{\mu_j, \xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x, q_j).$$
We have obtained this correction term out of a rough analysis to what is happening away from the blow-up points. Let us now analyze the region near them. That will allow us to identify the function $\mu_0(t)$ and the constants $b_j$. It is convenient to write

$$S(u) := -u_t + \Delta_x u + u^p.$$  

We consider the error of approximation $S(u_0)$. We have

$$S(u_{\mu,\xi}) = -\sum_{i=1}^{k} \partial_t u_i + \left( \sum_{i=1}^{k} u_i \right)^{p} - \sum_{i=1}^{k} U_{\mu_i,\xi_i}^{p}.$$
We obtain the following estimate near a given concentration point $q_j$, from where the formal asymptotic derivation of the unknown parameters will be a rather direct consequence.

Given $j$, assuming that

$$|x - q_j| \leq \frac{1}{2} \min_{i \neq l} |q_i - q_l|$$

and setting

$$y := \frac{x - \xi_j}{\mu_j},$$

we have

$$S(u_{\mu,\xi}) = \mu_j^{-\frac{n+2}{2}} \left( \mu_j E_{0j} + \mu_j E_{1j} + R_j \right)$$
\[ E_{0j} = pU(y)^{p-1} \left[ - \mu_j^{n-3} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{n-4} \mu_i^{n-2} G(q_j, q_i) \right] + \]

\[ \dot{\mu}_j \left[ y \cdot \nabla U(y) + \frac{n-2}{2} U(y) \right], \]

and

\[ E_{1j} = pU(y)^{p-1} \left[ - \mu_j^{n-2} \nabla_x H(q_j, q_j) + \right. \]

\[ \sum_{i \neq j} \mu_j^{n-2} \mu_i^{n-2} \nabla_x G(q_j, q_i) \left. \right] \cdot y + \dot{\xi}_j \cdot \nabla U(y) \]

and \( R_j \) contains smaller order terms.
The choice of the parameters at main order

We look for a solution of \((P)\) of the form

\[ u(x, t) = u_{\mu,\xi}(x, t) + \tilde{\phi}(x, t) \]

where \(\tilde{\phi}\) is smaller away from the blow-up points. The equation \(S(u) = 0\) becomes

\[ 0 = -\partial_t \tilde{\phi} + \Delta_x \tilde{\phi} + pu_{\mu,\xi}^{p-1} \tilde{\phi} + S(u_{\mu,\xi}) + \tilde{N}_{\mu,\xi}(\tilde{\phi}) \]

where \(\tilde{N}_{\mu,\xi}(\tilde{\phi}) = (u_{\mu,\xi} + \tilde{\phi})^p - u_{\mu,\xi}^p - pu_{\mu,\xi}^{p-1} \tilde{\phi}.\)
Neglecting the quadratic term, we want to approximately solve

\[-\partial_t \tilde{\phi} + \Delta_x \tilde{\phi} + p u_{\mu,\xi}^{p-1} \tilde{\phi} + S(u_{\mu,\xi}) = 0.\]

Near \( q_j \) we express \( \tilde{\phi} \) in terms of the slow variable \( y \),

\[
\tilde{\phi}(x, t) = \mu_j \frac{n-2}{2} \phi \left( \frac{x - \xi_j}{\mu_j}, t \right)
\]

and get the equation

\[-\mu_j^2 \partial_t \phi + \Delta_y \phi + pU(y)^{p-1} \phi + \mu_j \frac{n+2}{2} S(u_{\mu,\xi}) = 0\]

It is reasonable to assume that \( \phi(y, t) \) decays in the \( y \) variable.
Considering the largest term $E_0$ in the expansion of the error

$$\mu_j \frac{n+2}{2} S(u_{\mu, \xi})$$

we find that $\phi(y, t)$ should equal at main order a solution $\phi_0(y, t)$ of the elliptic equation

$$\Delta_y \phi_0 + pU^{p-1}\phi_0 = -\mu_j E_0 j \quad \text{in } \mathbb{R}^n, \quad \phi_0(y, t) \to 0 \quad \text{as } |y| \to \infty.$$ 

where we recall

$$E_0 j = pU(y)^{p-1} \left[ -\mu_j^{n-3} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-4}{2}} \mu_i^{\frac{n-2}{2}} G(q_j, q_i) \right] +$$

$$\dot{\mu}_j \left[ y_j \cdot \nabla U(y) + \frac{n-2}{2} U(y) \right],$$

When is this problem solvable?
We consider the linear problem

\[ L_0(\psi) := \Delta y \psi + p U^{p-1} \psi = h(y) \quad \text{in } \mathbb{R}^n, \quad \psi(y) \to 0 \quad \text{as } |y| \to \infty. \]

All bounded solutions of \( L_0(\psi) = 0 \) in \( \mathbb{R}^n \) consist of linear combinations of the functions

\[ Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \ldots, n, \quad Z_{n+1}(y) := \frac{n-2}{2} U(y) + y \cdot \nabla U(y). \]

If \( h(y) = O(|y|^{-m}), \; m > 2 \), then the problem is solvable iff

\[ \int_{\mathbb{R}^n} h(y) Z_i(y) \, dy = 0 \quad \text{for all } \; i = 1, \ldots, n + 1. \]
Since $n \geq 5$, we can solve

$$\Delta_y \phi_0 + pU^{p-1}\phi_0 = -\mu_0 j E_{0j} \quad \text{in } \mathbb{R}^n, \quad \phi_0(y, t) \to 0 \quad \text{as } |y| \to \infty.$$ 

provided that

$$\int_{\mathbb{R}^n} E_{0j}(y, t) Z_{n+1}(y) \, dy = 0 \quad \text{for all } j = 1, \ldots, k.$$
We compute

\[
\int_{\mathbb{R}^n} E_{0j}(y, t) Z_{n+1}(y) \, dy = c_1 \left[ \mu_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} \mu_j^{-2} \mu_i^{-2} G(q_i, q_j) \right] + c_2
\]

where \( c_1 \) and \( c_2 \) are the positive constants given by

\[
c_1 = -p \int_{\mathbb{R}^n} U^{p-1} Z_{n+1} = \frac{n-2}{2} \int_{\mathbb{R}^n} U^p, \quad c_2 = \int_{\mathbb{R}^n} |Z_{n+1}|^2.
\]

We observe that \( c_2 < +\infty \) thanks to the assumed fact \( n \geq 5 \).
These relations define a nonlinear system of ODEs for which a solution can be found as follows: we write

$$\mu_j(t) = b_j \mu_0(t)$$

and arrive at the relations

$$b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{n-2} G(q_i, q_j) + c_2 c_1^{-1} b_j^2 \mu_0^{3-n} \dot{\mu}_0(t) = 0$$
so that $\mu_0^{3-n} \dot{\mu}_0(t)$ should equal a constant, which is necessarily negative since $\mu_0$ decays to zero. This constant can be scaled out, hence it can be chosen arbitrarily to the expense of changing accordingly the values $b_i$. We impose

$$\dot{\mu}_0 = -\frac{2c_1 c_2^{-1}}{n-2} \mu_0^{n-3},$$

which yields after a suitable translation of time,

$$\mu_0(t) = \gamma_n t^{-\frac{1}{n-4}}, \quad \gamma_n = (2^{-1}(n-4)^{-1}(n-2)c_1^{-1}c_2)^{\frac{1}{n-4}}$$
and therefore the positive constants $b_j$ (in case they exist) must solve the nonlinear system of equations

$$b_j^{n-3} H(q_j, q_j) - \sum_{i \neq j} b_i^{\frac{n-2}{2}} b_j^{\frac{n-2}{2} - 1} G(q_i, q_j) = \frac{2b_j}{n - 2} \quad \text{for all} \quad j = 1, \ldots, k. \tag{S}$$

This system has a solution (which is unique) if the matrix $G(q)$ defined in is positive definite.
System (S) can be written as a variational problem. Indeed, it is equivalent to \( \nabla_b I(b) = 0 \) where

\[
I(b) := \frac{1}{n-2} \left[ \sum_{j=1}^{k} b_j^{n-2} H(q_j, q_j) - \sum_{i \neq j} b_i^{n-2} b_j^{n-2} G(q_i, q_j) - \sum_{j=1}^{k} b_j^2 \right]
\]

Writing \( \Lambda_j = b_j^{n-2} \) the functional becomes

\[
(n-2) I(b) = \tilde{I}(\Lambda) = \sum_{j=1}^{k} H(q_j, q_j) \Lambda_j^2 - \sum_{i \neq j} G(q_i, q_j) \Lambda_i \Lambda_j - \sum_{j=1}^{k} \Lambda_j^{4/n-2}.
\]
Let us assume that the matrix $G(q)$ is positive definite. Then the functional $\tilde{I}(\Lambda)$ is strictly convex in the region where all $\Lambda_j > 0$. It clearly has a global minimizer with all components positive. This yields the existence of a unique critical point $b$ of $I(b)$ with positive components which what we needed.
We fix $\mu_0(t)$ and the constants $b_j$ as above and let $\phi_{0j}$ solve for $\mu = \mu_0 = b_j \mu_0$

$$\Delta \phi_{0j} + pU(y)^{p-1} \phi_{0j} = -\mu_0 E_{0j}, \quad \phi_{0j}(y, t) \to 0 \quad \text{as } y \to \infty.$$ 

This leads us to the corrected ansatz,

$$u_{\xi, \mu}^*(x, t) := \sum_{j=1}^{k} U_{\mu_j, \xi_j}(x) - \mu_j^{\frac{n-2}{2}} H(x, q_j) + \mu_j^{\frac{n-2}{2}} \phi_{0j} \left( \frac{x - \xi_j}{\mu_j}, t \right).$$
The complete construction consists of writing

\[ u(x, t) = u_{\xi, \mu}^*(x, t) + \psi(x, t) + \sum_{j=1}^{k} \eta_j \mu_j \frac{-n-2}{2} \phi_j \left( \frac{x - \xi_j}{\mu_j}, t \right). \]

with \( \eta_j \) a cut-off on a large region on the slow variable \( y = \frac{x - \xi_j}{\mu_j} \), and

\[ \xi_j(t) = q_j + \zeta_j(t), \quad \mu_j(t) = b_j \mu_0(t) + \lambda_j(t) \]

with \( \zeta_j \) and \( \lambda_j \) parameters to be adjusted of order \( \mu_0^2 \).

- the functions \( \psi, \phi_1, \ldots, \phi_k \) satisfy a coupled system of outer-inner equations.
• The equation for $\psi$ (outer) is just a perturbation of a heat equation coupled with a linear term in the $\phi_j$.

• The (inner) problem for $\phi_j$ has the form

$$\mu_j^2 \partial_t \phi_j = L_0(\phi_j) + E_j(\zeta, \lambda)$$

in a large ball in slow-variable $y$.

• We are able to solve for a space decaying $\phi_j$ if $E_j$ satisfies orthogonality conditions that amount to a system of ODEs for $\zeta, \lambda$. Space decay is essential to make the coupling weak and then the full system solvable.
Finding an operator that solves the problem in a ball under the orthogonality conditions is a delicate step. The initial condition cannot be taken zero. To explain this:

$L_0$ has a positive radially symmetric bounded eigenfunction $Z_0$ associated to the only negative eigenvalue $\lambda_0$ to the problem

$$L_0(\phi) + \lambda \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).$$

Furthermore, $\lambda_0$ is simple and $Z_0$ decays like

$$Z_0(y) \sim |y|^{-\frac{n-1}{2}} e^{-\sqrt{|\lambda_0|} |y|} \quad \text{as} \quad |y| \to \infty.$$

Let $e(t) := \int_{\mathbb{R}^n} \phi(y, t)Z_0(y) \, dy$ where $\phi(y, t)$ solves

$$b_j^2 \mu_0^2 \partial_t \phi = L_0(\phi) + E(y, t)$$
Integrating against $Z_0$ in $\mathbb{R}^n$, using that $\mu_0(t)^2 \sim t^{-\frac{2}{n-4}}$ we get for some $b > 0$

$$bt^{-\frac{n-2}{n-4}} \dot{e}(t) - |\lambda_0|e(t) = f(t) := c \int_{\mathbb{R}^n} E(y, t)Z_0(y)\, dy.$$ 

Hence, for some $a > 0$,$$
e(t) = \exp(at^{\frac{n-2}{n-4}}) \left( e(t_0) + \int_{t_0}^{t} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}}) \, ds \right).$$

The only way in which $e(t)$ does not grow exponentially in time (and hence $\phi(y, t)$ remains bounded) is for the specific value

$$e(t_0) = \int_{\mathbb{R}^n} \phi(y, t_0)Z_0(y)\, dy = -\int_{t_0}^{\infty} s^{\frac{2}{n-4}} f(s) \exp(-as^{\frac{n-2}{n-4}})\, ds.$$
This argument suggests that the (small) initial condition required for $\phi$ should lie on a certain manifold locally described as a translation of the hyperplane orthogonal to $Z_0(y)$.

Since we have $k$ of these hyperplanes, these constraints define a codimension $k$ manifold of initial conditions which describes those for which the expected asymptotic bubbling behavior is possible.
Infinite-time bubbling in entire space turns out to be more delicate.

\[
\begin{cases}
  u_t = \Delta u + |u|^\frac{4}{n-2} u & \text{in} & \mathbb{R}^n \times (0, \infty), \\
  u(\cdot, 0) = u_0 & \text{in} & \mathbb{R}^n.
\end{cases}
\]

(P0)

It is expected to be highly sensitive to dimension and behavior of the initial datum \( u_0 \).

In a very interesting paper Fila and King consider a radially symmetric, positive \( u_0 \) with a exact power decay. Formal analysis, led them to conjecture that infinite time blow-up should only happen in low dimensions 3 and 4.
We rigorously establish the existence of solutions with infinite time blow-up in dimension 3, confirming the conjecture. Thus we consider

\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u + u^5 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\
    u(\cdot, 0) &= u_0 \quad \text{in} \quad \mathbb{R}^3.
\end{cases}
\end{align*}
\]

for a radial \( u_0 \) such that

\[
\lim_{|x| \to \infty} |x|^{\gamma} u_0(x) =: A > 0, \quad (G)
\]

We assume that \( \gamma > 1 \) which means that \( u_0(x) \) decays faster than the bubble \( U(x) \)
Theorem (del Pino, Musso, Wei, 2017)

Given $\gamma > 1$ there exists a positive, radially symmetric global solution $u(x, t)$ to problem \((P0)\) whose initial condition satisfies \((G)\) and as $t \to +\infty$ it has the blow up near the origin like

$$u(x, t) \sim \frac{1}{\mu(t)^{\frac{1}{2}}} U \left( \frac{x}{\mu(t)} \right),$$

$$\mu(t) \sim \begin{cases} 
  t^{1-\gamma} & \text{if} \quad 1 < \gamma < 2, \\
  t^{-1} \ln^2 t & \text{if} \quad \gamma = 2, \\
  t^{-1} & \text{if} \quad \gamma > 2.
\end{cases}$$
• The above form is valid in the inner self-similar region, $|x| \ll \sqrt{t}$, In the outer self-similar region $|x| \gg \sqrt{t}$, the solution dissipates in the form of a self-similar solution of heat equation

$$u_t = \Delta u \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty).$$

• A surprising feature of the result is the dynamics discovered for the blow-up rate $\mu(t)$. It is not quite an ODE: it has a highly non-local character governed by a nonlocal differential equation involving the fractional $\frac{1}{2}$-Caputo derivative.
• Dimension $n = 3$ in the case of a bounded domain. Strong connection with the Brezis-Nirenberg problem (1983)

$$\Delta u + \lambda u + u^5 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$ 

There exists a number $\lambda_\ast > 0$, the Brezis-Nirenberg number, such that a least energy positive solution exists whenever $\lambda_\ast < \lambda < \lambda_1(\Omega)$.

• When $\Omega$ is a ball $\lambda_\ast = \frac{\lambda_1}{4}$. 
In general (conjecture by Brezis-Peletier, proof by Druet 2003):
For $0 < \lambda < \lambda_1$

$$\Delta_x G_\lambda(x, y) + \lambda G_\lambda(x, y) + \gamma \delta_y(x) \text{ in } \Omega, \quad G(\cdot, y) = 0 \text{ on } \partial \Omega$$

$$G_\lambda(x, y) = \frac{\alpha_3}{|x - y|} - H_\lambda(x, y).$$

Then

$$\lambda_* = \sup \left\{ 0 < \lambda < \lambda_1 / \min_{\Omega} H_\lambda(\xi, \xi) > 0 \right\}.$$
What formally comes out: for a given point $q \in \Omega$, let

$$\lambda_* (q) = \sup \{ 0 < \lambda < \lambda_1 / H_\lambda(q, q) > 0 \}$$

(so that $H_{\lambda_*} (q, q) = 0$). Then a bubbling solution $u(x, t)$ of $(P)$ should have the approximate profile

$$u(x, t) \sim U_{\mu(t)}(x - q) - \mu(t)^{\frac{1}{2}} H_{\lambda_*}(x, q)$$

with

$$\mu(t) \sim e^{-2\lambda_* t}.$$ 

If $\Omega = B_1(0)$, $q = 0$, we have $\lambda_* = \frac{\pi^2}{4}$. 
Let $q = (q_1,\ldots,q_k)$ be such that the matrix $G_{\lambda}(q)$ is positive, where

$$G_{\lambda}(q) = \begin{bmatrix}
H_{\lambda}(q_1,q_1) & -G_{\lambda}(q_1,q_2) & \cdots & -G_{\lambda}(q_1,q_k) \\
-G_{\lambda}(q_1,q_2) & H_{\lambda}(q_2,q_2) & \cdots & -G_{\lambda}(q_2,q_k) \\
& \ddots & \ddots & \ddots \\
-G_{\lambda}(q_1,q_k) & \cdots & -G_{\lambda}(q_k-1,q_k) & H_{\lambda}(q_k,q_k)
\end{bmatrix}$$

Let

$$\lambda^*_*(q) = \sup \{0 < \lambda < \lambda_1 / G_{\lambda}(q) \text{ is positive definite}\}.$$

Then we conjecture the existence of a $k$-bubbling solution $u(x,t)$ to problem $(P)$ which away from the $q_j$ looks like

$$u(x,t) \approx \sum_{j=1}^{k} \mu_j(t)^{\frac{1}{2}} G_{\lambda^*_*}(x,q_j), \quad \mu(t) \sim b_j e^{-2\gamma^*_* t}$$
Thanks for your attention