A brief historical perspective on financial mathematics and some recent developments

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Brief history of mathematical finance: Early period

- The thesis of Louis Bachelier (1900) on the "Theory of Speculation"
- Introduction of Brownian motion to model fluctuating prices in the Paris stock exchange:

$$X(t_{n+1}) = X(t_n) + \mu \Delta t + \sigma \sqrt{\Delta t} W_{n+1}$$

  where $\{W_n\}$ are independent Gaussian random variables with mean zero and variance one (μ is mean return, σ is volatility, and $\sqrt{\Delta t} W_n$ are discrete Brownian increments)
- This is five years before Einstein's (1905) famous paper where the connection with diffusion theory (PDE) and molecular physics was made (Investigations into the theory of Brownian motion)
- Bachelier was ahead of his time in that he understood the passage from random walks to diffusions and, in general, the importance of sample paths or what we call today "stochastic analysis" (M. Taqqu, 2001)
- His work did not get much attention until the sixties, by P. Samuelson.

Brief history: The nineteen thirties and its tail

- Attempts to control speculation, which was widely believed to have caused the crash of 1929
- Dominance of "qualitative" macroeconomics (Keynes 1936 "General Theory")
- Some quantitative, ODE studies of business cycles and the role of government intervention (Goodwin 1967, Minsky 1982, Keen 1995)
- Dynamic Stochastic General Equilibrium (DSGE), currently a very actively pursued area of research in macroeconomics (the "neo-Keynsians"), in economics departments and central banks

Brief history: The Black-Scholes-Merton theory of option pricing, 1973

- By 1971 the Bretton Woods agreement (July 1944) of fixed currency exchange rates had created imbalances due to inflationary pressures in the US. It was abolished in August 1971 by President Nixon
- Fluctuating exchange rates created price volatility in international trade, partly agrevated by currency speculation. This was a shock to industry and to the world of economics, and led to Tobin's tax proposals in 1972 (the Tobin tax on currency transactions to limit speculation)

Brief history: The Black-Scholes-Merton theory of option pricing, 1973; continued

- A different, less interventionist and more market oriented group attempted to provide financial instruments that could manage this "new" risk. This gave birth to put and call options, that is, contracts whose payoff at a fixed time depends on the underlying exchange rate. They provided "insurance" against adverse fluctuations. But how should they be priced?
- Major breakthrough: Risk-neutral pricing using dynamically adjusted hedging portfolios, which is the BSM theory. Elegant presentation of these developments by P. Samuelson (SIAM Review, 1973). A highly original idea and a shift from the orthodoxy of supply-demand pricing (Nobel 1997). Shift to the role of the issuer of the option and not its buyer.
- Dramatic change in mathematical methodology: Enter stochastic calculus (Ito's formula, 1948, in Merton's 1969 MIT thesis) which until that time was an esoteric part of modern probability that was completely ignored by science and engineering (and still is, outside finance). Bachelier is back.

Brief history: The 1987 crash and stochastic volatility

- On October 19, 1987, the S&P500 index dropped 22.6%, widely believed to have been caused by "program trading"
- Spotlight falls on the role of volatility, for σ is not "constant" as assumed in the BSM price model, $\frac{dX(t)}{X(t)} = \mu dt + \sigma dW(t)$
- Volatility fluctuates randomly in a complicated way that affects option prices (as risk control or portfolio insurance instruments).
- Maintaining hedging portfolios, trading to hedge the risk of writing an option, increases volatility of the underlying stock prices (Frey-Stremme '98, Sircar-Papanicolaou '98, Schönbucher- Wilmott '00)
- Massive increase in complexity of the mathematical models in '90s (but not really used until recently): Computational methods enter derivative pricing, complicated time series estimation problems emerge (GARCH models)
- Feedback of trading to hedge (program trading) on prices leads to new types of problems in stochastic analysis (forward-backward SDE) that is a rapidly advancing research area today

Brief history: The golden age of fin math, 1987-2008

- By 2000 the financial mathematics research community understood well that mitigating the individual risk of transactions (using options) has significant macroeconomic consequences, such as increase in market volatility, and possibly systemic risk
- We also knew that trading to maintain an optimal investment portfolio (Merton theory) tends to decrease volatility and hence decreases systemic risk (an effect analyzed in Nayak and Papanicolaou '08). Therefore, not all trading is "destabilizing"
- The introduction of credit derivatives (credit default swaps) as instruments that manage risk due to interest rate fluctuations and default amplified systemic risk in ways that are very difficult to analyze and quantify. There were also unexpected effects of the wave of deregulations in late '90s
- At the hight of the credit derivative boom (2004-2008) the prices of these instruments, which are not being traded on exchanges like most equity options, were based on models that were widely off by any measure one chose to use. The theoreticians knew this well

What is systemic risk in finance?

- A relatively recent (mid-nineties) terminology meaning risk to a broader, many-agent financial or other system coming from positions and interactions (or transactions) that can change suddenly and rapidly when seemingly small changes have occurred to individual agents or among signals being tracked. An instability. A phase transition.
- Systemic risk, or instabilities, occur in many complex systems: In ecology (diversity of species), in climate change, in material behavior (phase transitions), etc. Mathematical methodologies do overlap (Haldane 2009, May 2010, US National Academy of Sciences BMSA Report 2007)

Why systemic risk now?

- The 2008 financial crisis. The role of mathematicians in the financial services industry. The teaching of financial mathematics.
- Recurring question: How do we quantify risk (risk measures) and systemic risk?
- Look at the role of market volatility (and liquidity) and transactions that enhance or suppress it.
- Systemic risk in statistical arbitrage: How seemingly benign, and intelligent, trading strategies can amplify systemic risk. Large fraction of transaction volume (including liquidity providers) today is from statistical arbitrage.
- Mathematics research in systemic risk today: Time dependent random matrix theory, large deviations, phase transitions, mean field games, ...

What are some sources of systemic risk today?

- Two types of trading in equities are widely practiced today: High-frequency (limit-order and market) trading and statistical arbitrage or market neutral (generalized) pairs trading
- These types of trading account for well over two thirds the volume traded today
- It is not yet clear how to quantify the systemic risk, or the market instabilities generated by these types of trading
- A lot of internal activity of hedge funds (in and out of banks) is focused on designing and implementing such trading strategies

What is Statistical Arbitrage?

A modern version of large-scale, data driven, generalized pairs trading:

- If two stocks with prices P_t and Q_t are correlated (two energy stocks, two banking stocks, etc), then we can regress one on the other
$$\frac{dP_t}{P_t} = \alpha dt + \beta \frac{dQ_t}{Q_t} + dX_t$$
where the trend α is usually small, and neglected, β is to be estimated, and X_t is the residual
- Now the trading strategy is to buy and hold P and to short an amount of Q if X is low, and do the reverse if X is high.
- The main point of this contrarian strategy is: "buy low and sell high" **based on relative price fluctuations**, which reflect temporary, relative overpricing or underpricing, rather than on "fundamentals".

Can this cause systemic risk?

- Pairs trading is **a bet on mean reversion** (A. Lo, early nineties). That is, it is essential that the residual X_t will oscillate with statistical regularity for otherwise pairs trading will fail and lead to loss.
- Successful pairs traders monitor mean reversion very carefully and close their positions as soon as there is any indication of "anomaly". It is not clear what effect this trading has on the markets themselves.
- We see that it is not just "increase in volatility" that tends to destabilize markets

Modern statistical arbitrage

Consider the daily returns of the whole US equities market of $N (\approx 1400)$ stocks (with capitalization above one billion) over a period of M days (252 days, one financial year)

$$R_{ik} = \frac{S_{i(t_0-(k-1)\Delta t)} - S_{i(t_0-k\Delta t)}}{S_{t_0-k\Delta t}}, \ i = 1, ..., N, \ k = 1, ..., M$$

Introduce normalized returns

$$Y_{ik} = \frac{R_{ik} - \bar{R}_i}{\bar{\sigma}_i}, \ \bar{R}_i = \frac{1}{M}\sum_{k=1}^{M} R_{ik}, \ \bar{\sigma}_i^2 = \frac{1}{M-1}\sum_{k=1}^{M}(R_{ik} - \bar{R}_i)^2$$

and form the normalized empirical covariance matrix $(N \times N)$

$$\rho = \rho(t_0) = YY^T$$

It is a symmetric, positive definite (random) matrix with eigenvalues

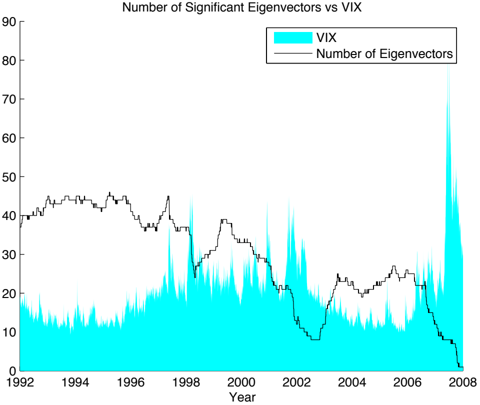
$$N \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N \geqslant 0$$

.

What does the empirical covariance tell us?

- The first eigenvalue λ_1 is significantly larger than the others, with a corresponding eigenvector $v^{(1)} = (v_1^{(1)}, v_2^{(1)}, \ldots, v_N^{(1)})$ that is essentially proportional to market capitalization. This is the "market" eigenspace. "Correlations discover capitalization".
- Subsequent eigenvectors have positive and negative entries, as they must, which can be associated with and used in generalized pairs trading portfolios.
- Eventually the eigenvalues drop to a noise level, close to zero, and the components of the eigenvectors oscillate. What is the dimension of the effective information subspace? The covariance projected onto the noise subspace behaves like that of a canonical random matrix with approximately a Marchenko-Pastur (1967) law for the distribution of its eigenvalues (Bouchaud-Potters-Laloux (2005), Avellaneda-Lee (2009)).

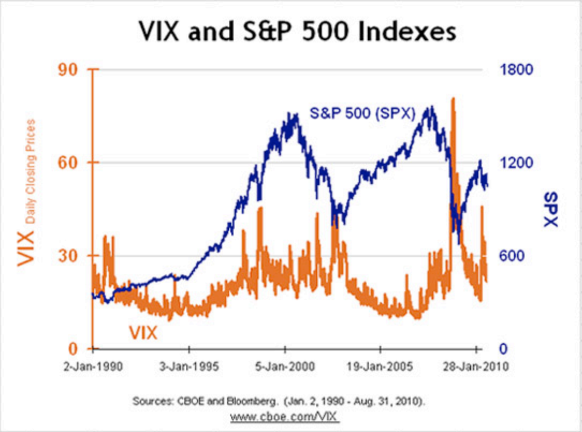
The dimension of the information subspace and the VIX



Market diversity (liquidity): The number of significant eigenvectors needed to explain the variance of the correlation matrix at the 55% level, from Jan 1992 to Dec 2008. The estimation window for the correlation matrix is 252 days. The boundary of the shaded region represents the VIX CBOE Volatility Index (by T. Callaghan and N. West).

S&P500 and the VIX

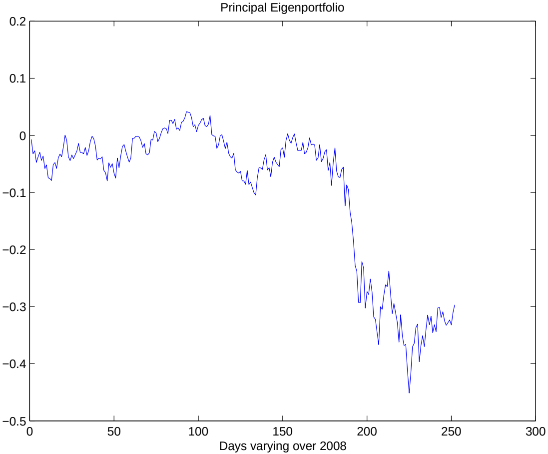
Market "temperature", the VIX, and the S&P500 move in opposite directions.



What does the empirical covariance tell us?

- Principal component analysis (PCA) orders the market into industry sectors consistent with exchange traded funds (ETF)s for these sectors!
- Market diversity (liquidity) and implied volatility (VIX) from options markets, computed from entirely different data sets, track each other!
- Eigenportfolios, or portfolios constructed using the market eigenvectors, have returns that are (strongly) dependent on market behavior.
- PCA suggests generalized pairs trading strategies that involve only the noise subspace and are therefore "decoupled" from the market (market neutral).

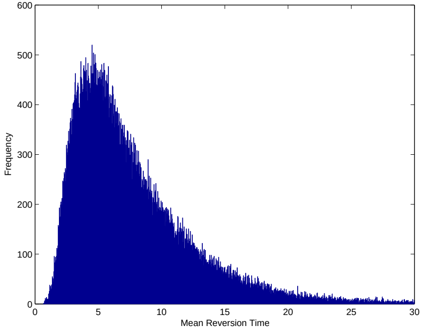
Index or principal eigenportfolio



Evolution of the principal eigenportfolio from Jan 2008 to Dec 2008 tracks the S&P500.

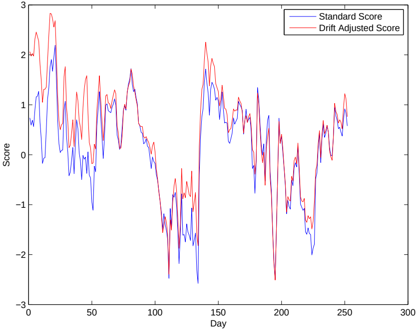
Betting on mean reversion

As random functions of time, are the residual or noise covariance components mean reverting?



Histogram of estimated Mean Reversion Times

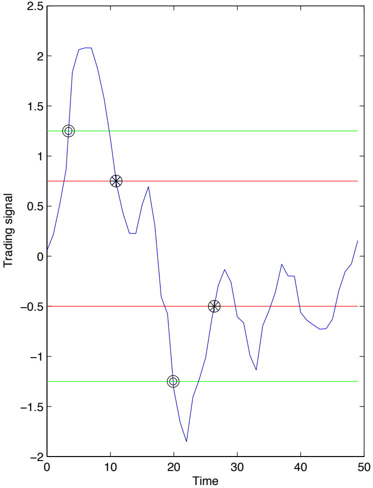
Trading signals for statistical arbitrage



Normalized by standard deviation signals or s(and drift-adjusted s_{d})-scores for Citigroup's Stock. Tendency for mean reversion is visible.

Basic statistical arbitrage strategy and mean reversion

The long-short (buy low-sell high) trading strategy for a "normalized" trading signal based on a noise subspace residual.

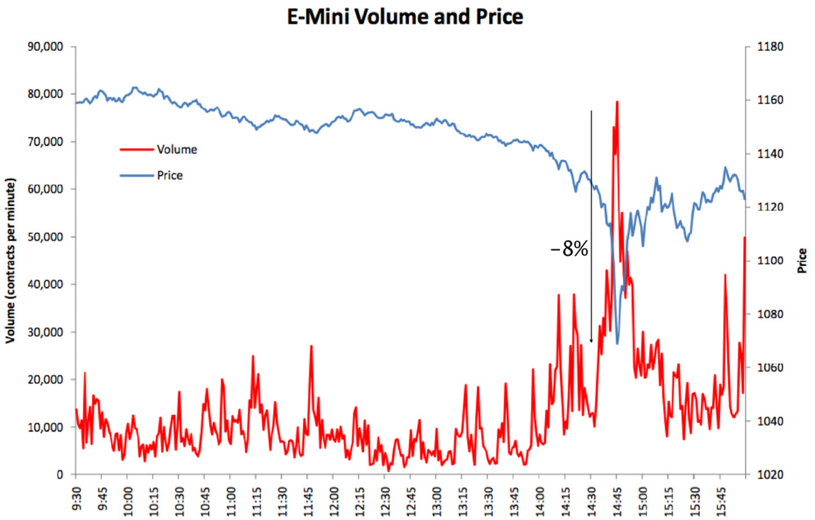


Systemic risk

What can go wrong? What are the emerging mathematical problems?

1. Since well calibrated mean reversion times are crucial in statistical arbitrage there are a lot of "unwind position" triggers built into the computer codes, reflecting the uncertainties inherent in the strategy.
2. Typical returns of well-run statistical arbitrage portfolios have Sharp ratios (excess return over volatility) of $1.0 - 1.5$, which are low, as expected. Therefore, large volumes and low transaction costs, with the use of ETFs, are very important. (Historical Sharp ratios for index portfolios are $3 - 4$ but for the last decade they were small, except in 2013.)
3. There is essentially no mathematical theory that can provide any insight into the fundamental premise of statistical arbitrage: That residuals are mean reverting with statistical regularity. The risks when betting on the market (index portfolios) are (thought to be) well understood. But the risks when betting on mean reversion are hidden and not clear at all (May 6, 2010?).

Flashcrash: SP500 on May 6, 2010



Concluding remarks

- Systemic risk is an inevitable consequence of the increased complexity, speed, sophistication, and globalization of finance. It is everywhere, but it is not easy to identify and to quantify. We are living the Information Revolution.
- High level academic financial mathematics research is placed exceptionally well so as to address systemic risk problems. It is not in the banks' interest to do it, the hedge funds are not interested in it, the governments do not know what to look for, and a lot of the current financial mathematics research is likely to stay close to more established areas, while CS will probably step in and capture the "trends" from the "data".
- The mathematics of systemic risk is truly challenging because while some useful methodology is already well developed (large deviations, phase transitions), much more diverse issues must be addressed, from large data sets to intricate interconnectivity and to uncertainty quantification that is needed at every step.