Kinetic integro-differential equations

Luis Silvestre

University of Chicago
What were we talking about?

**Parabolic integro-differential equations**

\[ u_t = \int (u(t, y) - u(t, x))K(t, x, y) \, dy. \]

Regularity results in the style of DeGiorgi/Nash (for divergence form equations) and Krylov & Safonov (for non-divergence form equations).

**The Boltzmann equation**

\[ f = f(t, x, v) \]

\[ f_t + v \cdot \nabla_x f = Q(f, f), \]

where \( Q(f, f) \) is a complicated nonlinear operator. It can be written as an integro-differential diffusion in the \( v \) variable with a very degenerate kernel \( K \) depending on \( f \).
**Divergence vs Non-divergence form operators**

**Divergence form operator.** \( u \mapsto \partial_i a_{ij}(x) \partial_j u. \)

It is a self adjoint operator with respect to \( L^2 \)

Integro-differentiable version:

\[
    u \mapsto \int (u(y) - u(x)) K(x, y) \, dy,
\]

with \( K(x, y) = K(y, x) \)

**Non-divergence form operator.** \( u \mapsto a_{ij}(x) \partial_{ij} u. \)

It gives us a bounded quantity if \( u \in C^2 \)

Integro-differentiable version:

\[
    u \mapsto \int (u(y) - u(x)) K(x, y) \, dy,
\]

with \( K(x, x + h) = K(x, x - h) \)
Divergence vs Non-divergence form operators

**Divergence form operator.** \( u \mapsto \partial_i a_{ij}(x) \partial_j u. \)

It is a self adjoint operator with respect to \( L^2 \)

Integro-differentiable version:

\[
u \mapsto \int (u(y) - u(x))K(x, y) \, dy,
\]

with \( K(x, y) = K(y, x) \)

**Non-divergence form operator.** \( u \mapsto a_{ij}(x) \partial_{ij} u. \)

Is gives us a bounded quantity if \( u \in C^2 \)

Integro-differentiable version:

\[
u \mapsto \int (u(y) - u(x))K(x, y) \, dy,
\]

with \( K(x, x + h) = K(x, x - h) \)
Divergence vs Non-divergence form operators

**Divergence form operator.** \( u \mapsto \partial_i a_{ij}(x) \partial_j u. \)

It is a self adjoint operator with respect to \( L^2 \)

Integro-differentiable version:

\[
 u \mapsto \int (u(y) - u(x))K(x, y) \, dy,
\]

with \( K(x, y) = K(y, x) \)

**Non-divergence form operator.** \( u \mapsto a_{ij}(x) \partial_{ij} u. \)

Is gives us a bounded quantity if \( u \in C^2 \)

Integro-differentiable version:

\[
 u \mapsto \int (u(y) - u(x))K(x, y) \, dy,
\]

with \( K(x, x + h) = K(x, x - h) \)
Divergence vs Non-divergence form operators

Divergence form operator. \( u \mapsto \partial_i a_{ij}(x) \partial_j u. \)

It is a self adjoint operator with respect to \( L^2 \)

Integro-differentiable version:

\[
  u \mapsto \int (u(y) - u(x)) K(x, y) \, dy,
\]

with \( K(x, y) = K(y, x) \)

Non-divergence form operator. \( u \mapsto a_{ij}(x) \partial_{ij} u. \)

Is gives us a bounded quantity if \( u \in C^2 \)

Integro-differentiable version:

\[
  u \mapsto \int (u(y) - u(x)) K(x, y) \, dy,
\]

with \( K(x, x + h) = K(x, x - h) \)
Divergence vs Non-divergence form operators

**Divergence form operator.** \[ u \mapsto \partial_i a_{ij}(x) \partial_j u. \]
It is a self adjoint operator with respect to \( L^2 \)
Integro-differentiable version:

\[ u \mapsto \int (u(y) - u(x))K(x, y) \, dy, \]

with \( K(x, y) = K(y, x) \)

**Non-divergence form operator.** \[ u \mapsto a_{ij}(x) \partial_{ij} u. \]
Is gives us a bounded quantity if \( u \in C^2 \)
Integro-differentiable version:

\[ u \mapsto \int (u(y) - u(x))K(x, y) \, dy, \]

with \( K(x, x + h) = K(x, x - h) \)
The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} (u(y) - u(x) - \mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)) K(x, y) \, dy.$$ 

- Upper bound: 
  $$K(x, y) \lesssim |x - y|^{-d-2s}.$$ 

- Ellipticity: 
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$ 

The term $\mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 
Non-divergence form integro-differential operators

The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} \left( u(y) - u(x) - \mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x) \right) K(x, y) \, dy.$$ 

- **Upper bound:**
  $$K(x, y) \lesssim |x - y|^{-d-2s}.$$ 

- **Ellipticity:**
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$ 

The term $\mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 
Non-divergence form integro-differential operators

The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} \left( u(y) - u(x) - \mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x) \right) K(x, y) \, dy.$$  

- **Upper bound:**
  $$K(x, y) \lesssim |x - y|^{-d-2s}.$$  

- **Ellipticity:**
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$  

The term $\mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 
The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} (u(y) - u(x) - 1_{B_1(y-x)} \nabla u(x) \cdot (y-x)) K(x, y) \, dy.$$  

- **Upper bound:**
  $$\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.$$  

- **Ellipticity:**
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$  

The term $1_{B_1(y-x)} \nabla u(x) \cdot (y-x)$ is required only if $s \geq 1/2$. 
Non-divergence form integro-differential operators

The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} (u(y) - u(x) - \mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)) K(x, y) \, dy.$$  

- **Upper bound:**
  $$\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.$$  

- **Ellipticity:**
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$  

The term $\mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 
Non-divergence form integro-differential operators

The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

$$Lu = \int_{\mathbb{R}^d} (u(y) - u(x) - 1_{B_1}(y - x) \nabla u(x) \cdot (y - x)) K(x, y) \, dy.$$ 

- Upper bound:

$$\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2 - 2s}.$$ 

- Ellipticity:

$$K(x, y) \gtrsim |x - y|^{-d - 2s}.$$ 

The term $1_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 
The operator $Lu$ should produce a bounded function when $u$ is $C^2$.

\[
Lu = \int_{\mathbb{R}^d} (u(y) - u(x) - \mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)) K(x, y) \, dy.
\]

- **Upper bound:**

\[
\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.
\]

- **Ellipticity:**

\[
\int_{B_r(x)} |e \cdot (x - y)|^2 K(x, y) \, dy \gtrsim r^{2-2s} \quad \text{for any } |e| = 1.
\]

The term $\mathbb{1}_{B_1}(y - x) \nabla u(x) \cdot (y - x)$ is required only if $s \geq 1/2$. 

Latest result in non-divergence form

Theorem (Schwab, S., 2014)

We make the following assumptions on the kernel $K \geq 0$.

- For every $r > 0$,
  \[
  \int_{B_r} |x - y|^{2-2s} K(t, x, y) \, dy \lesssim r^{2-2s}
  \]

- $K(t, x, y) \gtrsim |y - x|^{-d-2s}$ for all $y \in A_x$,
  where $A_x$ is symmetric and $|A_x \cap (B_{2r} \setminus B_r)| \geq \mu r^n$ for all $r > 0$.

Then the parabolic equation

\[
  u_t = \int_{\mathbb{R}^n} (u(t, x + y) - u(t, x))K(t, x, y) \, dy,
\]

satisfies the weak Harnack inequality and Hölder estimates.

Notes. Harnack inequality does not hold. The kernel $K$ may be non symmetric. There can be drift terms if $s \geq 1/2$. The estimates are uniform as $s \to 1$. 
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[ Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy. \]

- **Symmetry:** \(K(x, y) = K(y, x)\)
- **Upper bound** \((L : H^s \to H^{-s})\):
  \[ K(x, y) \lesssim |x - y|^{-d-2s}. \]

- **Coercivity** \((\langle Lu, u \rangle \gtrsim \|u\|^2_{H^s} + LOT)\):
  \[ K(x, y) \gtrsim |x - y|^{-d-2s}. \]
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[
Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy.
\]

- **Cancellation:**
  \[
  \int_{B_1(x)} K(x, y) - K(y, x) \, dy \lesssim 1.
  \]

- **Upper bound** \((L : H^s \to H^{-s})\):
  \[
  K(x, y) \lesssim |x - y|^{-d-2s}.
  \]

- **Coercivity** \((\langle Lu, u \rangle \gtrsim \|u\|_{H^s}^2 + LOT)\):
  \[
  K(x, y) \gtrsim |x - y|^{-d-2s}.
  \]
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[
Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy.
\]

- Cancellation:
  \[
  \int_{B_1(x)} K(x, y) - K(y, x) \, dy \lesssim 1.
  \]

- Upper bound (\(L : H^s \to H^{-s}\)):
  \[
  K(x, y) \lesssim |x - y|^{-d - 2s}.
  \]

- Coercivity (\(\langle Lu, u \rangle \gtrsim \|u\|_{H^s}^2 + LOT\)):
  \[
  K(x, y) \gtrsim |x - y|^{-d - 2s}.
  \]
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[
Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy.
\]

- Cancellation:

\[
\int_{B_1(x)} K(x, y) - K(y, x) \, dy \lesssim 1.
\]

- Upper bound \((L : H^s \to H^{-s})\):

\[
\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.
\]

- Coercivity \((\langle Lu, u \rangle \gtilde \|u\|_{H^s}^2 + LOT)\):

\[
K(x, y) \gtilde |x - y|^{-d-2s}.
\]
Divergence form integro-differential operators

The bilinear form $(u, v) \mapsto \langle Lu, v \rangle$ should be comparable with the inner product in $H^s$.

$$Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy.$$  

- **Cancellation:**
  $$\int_{B_{1}(x)} K(x, y) - K(y, x) \, dy \lesssim 1.$$  

- **Upper bound** ($L : H^s \to H^{-s}$):
  $$\int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.$$  

- **Coercivity** ($\langle Lu, u \rangle \gtrsim \| u \|_{H^s}^2 + LOT$):
  $$K(x, y) \gtrsim |x - y|^{-d-2s}.$$
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[ Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy. \]

- Cancellation:
  \[ \int_{B_1(x)} K(x, y) - K(y, x) \, dy \lesssim 1. \]

- Upper bound (\(L : H^s \to H^{-s}\)):
  \[ \int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}. \]

- Coercivity (\(\langle Lu, u \rangle \gtrsim \|u\|_{H^s}^2 + LOT\)):
  \[ K(x, y) \gtrsim |x - y|^{-d-2s}. \]
Divergence form integro-differential operators

The bilinear form \((u, v) \mapsto \langle Lu, v \rangle\) should be comparable with the inner product in \(H^s\).

\[
Lu = \int_{\mathbb{R}^d} (u(y) - u(x)) K(x, y) \, dy.
\]

- **Cancellation:**
  \[
  \int_{B_1(x)} K(x, y) - K(y, x) \, dy \lesssim 1.
  \]

- **Upper bound \((L : H^s \to H^{-s})\):**
  \[
  \int_{B_r(x)} |x - y|^2 K(x, y) \, dy \lesssim r^{2-2s}.
  \]

- **Coercivity \(\langle Lu, u \rangle \gtrsim \|u\|_{H^s}^2 + LOT\):**
  \[
  \int_{B_r(x)} |e \cdot (x - y)|^2 K(x, y) \, dy \gtrsim r^{2-2s} \text{ for any } |e| = 1.
  \]
## Kinetic diffusion equations

### Theorem

\[ f_t + v \cdot \nabla_x f - L_v f = g \quad \text{in } Q_1, \]

where \( g \in L^\infty \) and \( L_v \) is an integro-differential operator. Then

\[ \| f \|_{C^\alpha(Q_{1/2})} \lesssim \| f \|_{L^\infty(Q_1)} + \| g \|_{L^\infty(Q_1)}, \]

for some \( \alpha > 0 \). Here \( Q_r = (-r^{2s}, 0] \times B_{r^{1+2s}} \times B_r \).

The diffusion \( L_v f \) is in the \( v \) variable only!

**Most natural case:** \( L_v \) of non-divergence form.

**What we can do:** \( L_v \) of divergence form.

**Boltzmann case:** \( L_v \) satisfies the upper bound on average and the lower bound on cones. It has non-divergence symmetry. It satisfies the cancellation condition so that we can put it in divergence form too.
Hölder estimate for kinetic integro-differential equations

Theorem (Imbert, S., 2016)

Let \( f : [-1, 0] \times B_1 \times \mathbb{R}^d \rightarrow \mathbb{R} \) solve the equation

\[
    f_t + v \cdot \nabla_x f - \int_{\mathbb{R}^d} (f(t, x, v') - f(t, x, v))K(t, x, v, v') \, dv = g.
\]

For \( s \in (0, 1) \), assume,

Coercivity: \( \| f \|_{H^s}^2 \lesssim \iint |f(v') - f(v)|^2 K(v, v') \, dv' \, dv. \)

Upper bound: \( \int_{B_R(v)} K(v, v')|v - v'|^2 \, dv' \lesssim R^{2-2s} \)

Cancellation:
\[
    \int K(v, v') - K(v', v) \, dv' \lesssim 1, \quad \int (v - v')(K(v, v') - K(v', v)) \, dv' \lesssim 1
\]

Then \( f \) satisfies Hölder regularity estimates

\[
    \| f \|_{C^\alpha([-1/2, 0] \times B_{1/2} \times B_{1/2})} \lesssim \| f \|_{L^\infty} + \| g \|_{L^\infty}.
\]
Kolmogorov equation

For any $s \in (0, 1)$, the equation

$$f_t + \nabla_x f + (-\Delta)^s f = 0,$$

admits a smooth heat kernel.

Sharp regularity estimates are trivial by analysing this kernel and the scaling the equation.

In the case $s = 1$ the heat kernel was explicitly computed by Kolmogorov

$$H(t, x, v) = \frac{c}{t^{2d}} \exp \left( - \frac{|v|^2}{4t} - \frac{3|x - tv/2|^2}{t^3} \right).$$
Consider the equation

\[ f_t + \nu \cdot \nabla_x f + \partial_v a_{ij}(t, x, \nu) \partial_{v_j} f = g, \]

where \( f \in L^2 \) and \( g \in L^\infty \) and \( a_{ij} \) is uniformly elliptic. Then \( f \) is Hölder continuous.

- A. Pascucci & S. Polidoro [2004], \( \|f\|_{L^\infty(Q_{1/2})} \lesssim \|f\|_{L^2(Q_1)}. \)
- WD Wang & LQ Zhang [2009], \( \|f\|_{C^\alpha(Q_{1/2})} \lesssim \|f\|_{L^\infty(Q_1)}. \)
Classical proofs of Hölder estimates

▶ **Step 1.** $L^2 \rightarrow L^\infty$.

Let $f \geq 0$ be a supersolution: $f_t + \nu \cdot \nabla_x f - L_\nu f \geq 0$.

There exist $\varepsilon > 0$ and $\delta > 0$ so that

$$|\{f \geq 1\} \cap Q_1| \geq (1 - \varepsilon) Q_1 \implies f \geq \delta \text{ in } Q_{1/2}.$$ 

DG: energy dissipation $\rightarrow$ improvement of integrability $\rightarrow$ iteration.

K&S: Alexandroff estimate.

▶ **Step 2.** $L^\infty \rightarrow C^\alpha$.

Hölder continuity $\iff$ improvement of oscillation lemma

$\iff$ There exists $\delta > 0$ so that

$$|\{f \geq 1\} \cap Q_1| \geq \frac{1}{2} Q_1 \implies f \geq \delta \text{ in } Q_{1/2}.$$ 

DG: $f \in H^1 \Rightarrow$ lower bound on intermediate sets.

K&S: barrier functions + inkspots.
If $f \geq 0$ is a subsolution $f_t + \nu \cdot \nabla_x f - L_\nu f \leq 0$, multiplying times $f$ and integrating, we get

$$\sup_t \|f\|_{L^2_{x,v}} + \|f\|_{L^2_{t,x}(H^s_\nu)} \lesssim \|f\|_{L^2_{t,x,v}}.$$ 

No gain of regularity in $x$, thus no obvious gain of integrability.

Using heat kernel for the Kolmogorov equation (semi-explicit formula) we get a gain of integrability: $\|f\|_{L^p_{t,x,v}} \lesssim \|f\|_{L^2_{t,x,v}}$ for some $p > 2$. 

**Energy dissipation**
Energy dissipation

If $f \geq 0$ is a subsolution $f_t + \nu \cdot \nabla_x f - L_{\nu} f \leq 0$, multiplying times $f$ and integrating, we get

$$\sup_t \|f\|_{L^2_{x,\nu}} + \|f\|_{L^2_{t,x}(H^s_\nu)} \lesssim \|f\|_{L^2_{t,x,\nu}}.$$

Back to the equation. The function $f$ also solves

$$f_t + \nu \cdot \nabla_x f + (-\Delta)^s f = (-\Delta)^s f + L_{\nu} f - \mu,$$

where $\mu$ is a bounded positive measure, and $(-\Delta)^s f + L_{\nu} f$ is bounded in $L^2_{t,x}(H^{-s}_\nu)$.

Using heat kernel for the Kolmogorov equation (semi-explicit formula) we get a gain of integrability: $\|f\|_{L^p_{t,x,\nu}} \lesssim \|f\|_{L^2_{t,x,\nu}}$ for some $p > 2$. 

Energy dissipation

If $f \geq 0$ is a subsolution $f_t + v \cdot \nabla_x f - L_v f \leq 0$, multiplying times $f$ and integrating, we get

$$\sup_t \|f\|_{L^2_{x,v}} + \|f\|_{L^2_{t,x}(H^s_v)} \lesssim \|f\|_{L^2_{t,x,v}}.$$ 

Back to the equation. The function $f$ also solves

$$f_t + v \cdot \nabla_x f + (-\Delta)^s_v f = (-\Delta)^s_v f + L_v f - \mu,$$

where $\mu$ is a bounded positive measure, and $(-\Delta)^s_v f + L_v f$ is bounded in $L^2_{t,x}(H^{-s}_v)$.

Using heat kernel for the Kolmogorov equation (semi-explicit formula) we get a gain of integrability: $\|f\|_{L^p_{t,x,v}} \lesssim \|f\|_{L^2_{t,x,v}}$ for some $p > 2$. 
Second step: from $L^\infty$ to $C^\alpha$.

**Case** $s \geq 1/2$. It is possible to do a compactness argument to reproduce De Giorgi’s inequality.

Key fact: $f : B_1 \to \mathbb{R}$, $f = 1_A$ and $f \in H^s \Rightarrow f$ is constant.

**Case** $s < 1/2$. The equation is in non-divergence form!

We use barriers and the spreading ink-spots argument as in the proof of Krylov & Safonov.

However,

- Constructing useful barriers for $f_t + v \cdot \nabla_x f - L_v f = 0$ is rather involved.

- The ink-spots construction has to be adapted to account for the obliqueness in the $x$ variable.

**Boltzmann case.** The equation is in non-divergence form. The proof of the second case suffices for all values of $s \in (0, 1)$. 
Second step: from $L^\infty$ to $C^\alpha$.

**Case** $s \geq 1/2$. It is possible to do a compactness argument to reproduce De Giorgi’s inequality.

Key fact: $f : B_1 \to \mathbb{R}$, $f = 1_A$ and $f \in H^s \implies f$ is constant.

**Case** $s < 1/2$. The equation is in non-divergence form!

We use barriers and the spreading ink-spots argument as in the proof of Krylov & Safonov.

However,

- Constructing useful barriers for $f_t + v \cdot \nabla_x f - L_v f = 0$ is rather involved.
- The ink-spots construction has to be adapted to account for the obliqueness in the $x$ variable.

**Boltzmann case.** The equation is in non-divergence form. The proof of the second case suffices for all values of $s \in (0, 1)$. 
Second step: from $L^\infty$ to $C^\alpha$.

**Case** $s \geq 1/2$. It is possible to do a compactness argument to reproduce De Giorgi’s inequality.
Key fact: $f : B_1 \to \mathbb{R}$, $f = 1_A$ and $f \in H^s$, $\implies f$ is constant.

**Case** $s < 1/2$. The equation is in non-divergence form!
We use barriers and the spreading ink-spots argument as in the proof of Krylov & Safonov.
However,
- Constructing useful barriers for $f_t + v \cdot \nabla_x f - L_v f = 0$ is rather involved.
- The ink-spots construction has to be adapted to account for the obliqueness in the $x$ variable.

**Boltzmann case.** The equation is in non-divergence form. The proof of the second case suffices for all values of $s \in (0, 1)$. 
What’s next for Boltzmann?

We proved that the solution to the (inhomogeneous) Boltzmann equation without cutoff is bounded and locally Hölder continuous depending on hydrodynamic quantities only.

Questions for the future:

- Higher regularity. Schauder estimates for kinetic integro-differential equations?
- Decay for large velocities.
- Weaker assumptions on the hydrodynamic quantities?
- Unconditional regularity. Out of reach?
Open problems regarding kinetic equations

Are Hölder estimates valid for equations in non-divergence form?

2nd order case:

$$f_t + v \cdot \nabla_x f - a_{ij}(t, x, v) \partial_{v_i v_j} f = 0.$$ 

Harnack? Weak Harnack? Hölder estimates?

Integro-differential case:

$$f_t + v \cdot \nabla_x f - L_v f = 0.$$ 

where $L_v$ is an integro-differential operator with rough coefficients in non-divergence form.
Open problems for integro-differential equations with singular kernels

Are Hölder estimates valid for integro-differential equations with singular kernels?

\[ u_t - \int (u(t, y) - u(t, x))K(t, x, y) \, dy = 0, \]

where Either \( K(x, y) = K(y, x) \) or \( K(x, x + h) = K(x, x - h) \), and for every unit vector \( e \in \partial B_1, \, r > 0 \),

\[ \int_{B_r} |e \cdot (y - x)|^2 K(x, y) \, dy \approx r^{2-2s}. \]

Current results require stronger conditions regarding the lower bound of \( K(x, y) \).
Open problem regarding coercivity of bilinear forms

Assume that for every $e \in \partial B_1$, $r > 0$,

$$\int_{B_r} |e \cdot (y - x)|^2 K(x, y) \, dy \gtrsim r^{2-2s}.$$ 

Does it imply the following inequality?

$$\int_{B_{2R} \times B_{2R}} |u(x) - u(y)|^2 K(x, y) \, dy \, dx \gtrsim \int_{B_{R} \times B_{R}} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} \, dy \, dx$$

The implication with the inequality in the reverse order is true. The current proofs require stronger assumptions on $K$. 