Integro-differential equations and Boltzmann

Luis Silvestre

University of Chicago

Rivière - Fabes symposium
Nestor Riviere

Nestor Riviere

Ariel and Joana
Plan for the talks

Talk 1

- Gentle introduction to integro-differential equations
- Regularity results for nonlocal parabolic equations
- Intro to the Boltzmann equation
- Available regularity results
- Rewriting the Boltzmann equation
- Proof of the upper bound

Talk 2

- Understanding operators in div and non-div form
- Understanding the notion of ellipticity for integral equations
- Hölder estimates for kinetic parabolic equations
- Our result
- Future plans and open problems
Let us consider the following random walk.

We have a given initial point: $X_0 = x$. The next point $X_{i+1}$ is selected randomly in each step uniformly in the ball of radius one centered at $X_i$. Let $X_N$ be the first point outside of the domain $\Omega \subset \mathbb{R}^d$. 

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A jump process

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Given a function $f : \mathbb{R}^d \setminus \Omega \rightarrow \mathbb{R}$, let

$$u(x) = \mathbb{E}[f(X_N)].$$

The function $u$ satisfies the equation

$$u(x) = f(x), \quad x \in \mathbb{R}^d \setminus \Omega,$$

$$u(x) = \frac{1}{|B_1|} \int_{B_1(x)} u(y) \, dy, \quad x \in \Omega.$$
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Basic properties of the equation

\[ u(x) = f(x), \quad x \in \mathbb{R}^d \setminus \Omega, \]

\[ \int_{B_1(x)} (u(y) - u(x)) \, dy = 0, \quad x \in \Omega. \]

- The maximum principle holds,

\[ \max_{\Omega} u \leq \max_{\mathbb{R}^d \setminus \Omega} f. \]

- Uniqueness of solutions for each continuous \( f \)

- The solution \( u \) becomes more regular inside \( \Omega \).
Parabolic case

Let $X_t$ be a stochastic process. Now $t$ is a continuous variables. $X_t$ jumps by time to time following the same uniform distribution as before. The jump times are determined by a Poisson process of intensity $1$.

Let $f : \mathbb{R}^d \to \mathbb{R}$. We define

$$u(t, x) = \mathbb{E}[f(X_t)].$$

which satisfies,

$$u_t(t, x) = \frac{1}{|B_1|} \int_{B_1(x)} (u(y) - u(x)) \, dy.$$
Parabolic case

Let $X_t$ be a stochastic process. Now $t$ is a continuous variables. $X_t$ jumps by time to time following a distribution $K$. The jump times are determined by a Poisson process of intensity $1$.

Let $f : \mathbb{R}^d \to \mathbb{R}$. We define

$$u(t, x) = \mathbb{E}[f(X_t)].$$

which satisfies,

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which satisfies,

$$u_t(t, x) = \int_{\mathbb{R}^d} (u(y) - u(x)) K(y - x) \, dy.$$

$$\int_{\mathbb{R}^d} K(y) \, dy = 1, \quad K \geq 0.$$
Parabolic case

Let $X_t$ be a stochastic process. Now $t$ is a continuous variables. $X_t$ jumps by time to time following a distribution $K$. The jump times are determined by a Poisson process of intensity $\kappa$.

Let $f : \mathbb{R}^d \to \mathbb{R}$. We define

$$u(t, x) = \mathbb{E}[f(X_t)].$$

which satisfies,

$$u_t(t, x) = \int_{\mathbb{R}^d} (u(y) - u(x))K(y - x) \, dy.$$

$$\int_{\mathbb{R}^d} K(y) \, dy = \kappa, \quad K \geq 0.$$
Parabolic case

Let \( X_t \) be a stochastic process. Now \( t \) is a continuous variables. \( X_t \) jumps by time to time following a distribution \( K \). The jump times are determined by a superposition of Poisson processes.

Let \( f : \mathbb{R}^d \to \mathbb{R} \). We define

\[
    u(t, x) = \mathbb{E}[f(X_t)].
\]

which satisfies,

\[
    u_t(t, x) = \int_{\mathbb{R}^d} (u(y) - u(x)) K(y - x) \, dy.
\]

\[
    \int_{\mathbb{R}^d} K(y) \, dy = +\infty, \quad K \geq 0.
\]
Classical elliptic operators

Classical elliptic operators are limits of integro-differential equations.

\[ \Delta u(x) = \lim_{r \to 0} \int_{\mathbb{R}^d} (u(y) - u(x)) K_r(y - x) \, dy, \]

where

\[ K_r(y) = \frac{c_n}{r^{n+2}} 1_{B_r}(y). \]

(and also for several different choices of \( K_r \))
Nonlocal parabolic equations

\[ u_t(t, x) = \int_{\mathbb{R}^d} (u(t, y) - u(t, x)) K(t, x, y) \, dy. \]

- \( K \geq 0 \implies \) maximum principle \( \implies \) uniqueness.
- Sometimes the equations has a regularization effect. Idea: the equation pushes the values of \( u \) at \( x \) to match the average of its neighbors.
- The solution will be more regular than the initial data whenever this effect is uniform at all scales.
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The fractional heat equation

\[ u_t(t, x) = \int_{\mathbb{R}^d} (u(t, y) - u(t, x)) |y - x|^{-d - 2s} \, dy. \]

Note that

\[ (-\triangle)^s u(x) = c_{n,s} \int_{\mathbb{R}^d} (u(x) - u(y)) |y - x|^{-d - 2s} \, dy. \]

\[ (-\triangle)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi). \]

The kernel \( K(y) = |y|^{-d - 2s} \) is not integrable. The integral is classical \( s < 1/2 \) and a principal value when \( 1/2 \leq s < 1 \).

An integro-differential operator is elliptic of order \( 2s \) when \( K(y) \approx |y|^{-d - 2s} \).
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An integro-differential operator is elliptic of order \( 2s \) when \( K(y) \approx |y|^{-d-2s} \).
Classical regularity results

Assume $\lambda I \leq \{a_{ij}(x, t)\} \leq \Lambda I$ point-wise.

Divergence form

**Theorem (De Giorgi, Nash and Moser. 1956-57)**

Let $u$ be a solution of $u_t = \partial_i a_{ij}(x, t) \partial_j u$ in $B_1 \times [-1, 0]$. Then

$$\|u\|_{C^\alpha(B_{1/2} \times [-1/2,0])} \leq C\|u\|_{L^2(B_1 \times [-1,0])}$$

Non divergence form

**Theorem (Krylov and Safonov. 1980)**

Let $u$ be a solution of $u_t = a_{ij}(x, t) \partial_{ij} u$ in $B_1 \times [-1, 0]$. Then

$$\|u\|_{C^\alpha(B_{1/2} \times [-1/2,0])} \leq C\|u\|_{L^\infty(B_1 \times [-1,0])}$$
Estimates for nonlocal equations in divergence form

They refer to the study of gradient flows of functionals in $H^s$.

A DeGiorgi-Nash-Moser type theorem gives us a Hölder estimates for an equation of the form

$$u_t = \int (u(y) - u(x)) K(x, y) \, dy.$$ 

which is the gradient flow of

$$F(u) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 K(x, y) \, dx \, dy.$$ 

Results by:
Komatsu, Z-Q Chen, Bass, Kassmann, Barlow, Caffarelli, Vasseur, C-H Chan, Kumagai, Kim, Song, Felsinger, Schwab, etc...
Assumptions for non local equations in divergence form

\[ u_t = \int_{\mathbb{R}^d} (u(t, y) - u(t, x))K(t, x, y) \, dy. \]

The essential assumption in order to obtain a DeGiorgi-Nash-Moser type result is

\[ \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 K(x, y) \, dx \, dy \approx \|u\|_{H^s}^2. \]

This is true in particular if

\[ K(x, y) \approx |x - y|^{-d-2s}, \]

\[ K(x, y) = K(y, x). \]
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But the result also holds under much more general assumptions. (Note. The Harnack inequality breaks)
Estimates for non local equations in non divergence form

They apply to equations of the form

\[ u_t = \int_{\mathbb{R}^d} (u(t, y) - u(t, x) - (y - x) \cdot \nabla u(t, x) 1_{B_1}(y - x)) K(t, x, y) \, dy. \]

Typical assumption: \( K(t, x, y) \approx |y - x|^{-d-2s}. \)

Partial history of results.

- Bass - Levin [2002] (probabilistic)
- Bass - Kassmann [2005] (probabilistic, variable order)
- Silvestre [2006] (analytic)
- Silvestre - Caffarelli [2009] (robust estimates)
- Bjorland - Caffarelli - Figalli [2012], Kassmann - Rang - Schwab [2013] and Silvestre-Schwab [2016] (singular kernels)
Aplicaciones

- **Discontinuous stochastic processes**
  - **Financial mathematics**: as in the book of R. Cont and P. Tankov.
  - **Physics**: survey articles by R. Metzler and J. Klafter.

- **Nonlocal electrostatics.** Applications to protein docking studied by a group in ZBI including A. Hildebrandt, R. Blossey, S. Rjasanow, O. Kohlbacher, H.P. Lenhof.

- **Image processing.** Including the work of S. Osher, P. Guidotti, etc...

- **PDEs of fluids.** For example the quasi-geostrophic equation or the Muskat problem.

- **Conformal Geometry.** Including the work of A. Chang, M. Gonzalez

- **The Dirichlet to Neumann map.** N. Guillen and R. Schwab use integral equations to study a the homogenization of the Neumann condition for an elliptic PDE.

- **The Boltzmann equation.**
The Boltzmann equation

The function $f(t,x,v)$ represents the density of gas particles.

$$f_t + v \cdot \nabla_x f = Q(f, f).$$

The Boltzmann collision operator is a nonlocal operator given by

$$Q(f, f)(v) = \int_{\mathbb{R}^n} \int_{S^{d-1}} \left( f(v') f(v') - f(v_*) f(v) \right) B(|v-v_*|, \theta) d\sigma dv_*.$$
The collision kernel

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There are several choices of collision kernels $B(v - v_*, \theta)$. In a physical model where particles interact by inverse power laws, we get

$$B(r, \theta) \approx r^\gamma |\theta|^{-d+1-2s},$$

with $\gamma > -d$, $s \in (0, 1)$. 
Macroscopic quantities

The mass, energy and momentum are conserved

\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, \nu) \, d\nu \, dx = M, \]
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, \nu) |\nu|^2 \, d\nu \, dx = E, \]
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, \nu) \nu \, d\nu \, dx = Q. \]

The entropy is non-increasing

\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, \nu) \log f(t, x, \nu) \, d\nu \, dx = H(t) \]
Simplified models

The space homogeneous model.
In this case we consider $f = f(t, v)$ independent of $x$. The equation does not have a drift term.

$$f_t = Q(f, f).$$

Grad’s cutoff assumption.
Consists in taking $B$ integrable.
We do not expect any regularization effect under this assumption.
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Consists in taking $B$ integrable.
We do not expect any regularization effect under this assumption.
Previous regularity estimates

All of the following results use the coercivity estimate as their main tool.

- Solutions of the space homogeneous Boltzmann equations become immediately $C^\infty$ for $\gamma + 2s \geq 0$: Desvillettes, Wennberg (2005), Alexandre, El Safadi (2005, 2009), Huo, Morimoto, Ukai, Yang (2008), Morimoto, Ukai, Xu, Yang (2009), Chen, He (2011).

- If the initial data is sufficiently close to equilibrium (in a weighted Sobolev space), then there is a global smooth solution: Gressman, Strain (2010, 2011), Alexandre, Morimoto, Ukai, Xu, Yang (2010, 2011).

- For nice initial data, there is a smooth solution for short time: Alexandre, Morimoto, Ukai, Xu, Yang (2010).

- The solution to the full Boltzmann equation is $C^\infty$ provided that it stays in $H^5$ (in all variables) plus infinite moments: Alexandre, Morimoto, Ukai, Xu, Yang (2010).
Hydrodynamic limit

Let $f^\varepsilon$ solve the equation

$$\partial_t f^\varepsilon + \nu \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon, f^\varepsilon).$$

Then, as $\varepsilon \to 0$, $f^\varepsilon$ converges to a function of the form $f(t, x, \nu) = a(t, x)e^{-b(t,x)|\nu-u(t,x)|^2}$, so that the hydrodynamic quantities

- mass density: $\rho(t, x) = \int f(t, x, \nu) \, d\nu$,
- velocity: $u(t, x) = \frac{1}{\rho(t, x)} \int \nu f(t, x, \nu) \, d\nu$,
- temperature: $\theta(t, x) = \frac{1}{3} \int |\nu - u(t, x)|^2 f(t, x, \nu) \, d\nu$.

solve the compressible Euler equation (which develops singularities in finite time).

Moreover, the entropy density converges to $\rho \log(\rho^{2/3}/\theta)$.
## Objectives and results

### Our program

<table>
<thead>
<tr>
<th>Conjecture</th>
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<tbody>
<tr>
<td>The solution of the Boltzmann equation, inhomogeneous and without cutoff, remains smooth for as long as hydrodynamic quantities stay bounded.</td>
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### Our results so far

<table>
<thead>
<tr>
<th>Theorem (S. (2016))</th>
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<tbody>
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<td>For $\gamma + 2s &gt; 0$, Let $f$ be a solution to the Boltzmann equation. Assuming controlled hydrodynamic quantities, then the solution $f$ satisfies $f(t, x, v) \lesssim 1 + t^{-\frac{d}{2s}}$</td>
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<th>Theorem (Imbert, S. (arXiv 1608.07571))</th>
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<td>Let $f$ be a bounded solution to the Boltzmann equation. Assuming controlled hydrodynamic quantities, then the solution $f$ is locally Hölder continuous.</td>
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The first chapter of Cedric Villani’s best seller *Théorème vivant* (English title: *Birth of a theorem*), describes an old conversation between the author and Clement Mouhot. Here are some extracts

- *My old deamon is back again: regularity for the inhomogeneous Boltzmann equation.*
- *I broke the problem down using a series of scale models, but even the simplest one baffles me.*
- *Let’s assume grazing collisions, okay? A model without cutoff. Then the equation behaves like a fractional diffusion, degenerate, of course, but a diffusion just the same, and as soon as you have bounds on density and temperature you can apply a Moser-style iteration scheme, modified to take nonlocality into account.*
- *Bernt tried a while ago, he gave up. A whole bunch of people have tried, but no one’s had any luck so far. Still, it’s plausible.*
The Boltzmann collision operator again

\[ Q(f, f)(v) = \int_{\mathbb{R}^n} \int_{S^{d-1}} \left( f(v'_*) f(v') - f(v'_*) f(v) \right) B(|v - v_*|, \theta) d\sigma dv_* . \]

\( K_f \) is a kernel depending on \( f \) and the cross section \( B \). It can be computed after a somewhat difficult change of variables in the integral.

\( b \) is a scalar function depending on the cross section \( B \) only. If \( B(r, \theta) \approx r^\gamma \theta^{-d+1-2s} \) then \( b(v) \approx |v|^\gamma \).
The Boltzmann collision operator again

\[ Q(f, f)(v) = \int_{\mathbb{R}^n} \int_{S^{d-1}} \left( f(v') f(v') - f(v') f(v) \right) B(|v - v_*|, \theta) \]
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The Boltzmann collision operator again

\[ Q(f, f)(v) = \int_{\mathbb{R}^n} \int_{S^{d-1}} \left( f(v') - f(v) \right) f(v_*) B(|v - v_*|, \theta) \]

\[ f(v) \left( f(v'_*) - f(v_*) \right) B(|v - v_*|, \theta) d\sigma dv_* . \]

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\[ + f(v) \int_{\mathbb{R}^n} \int_{S^{d-1}} \left( f(v_*) - f(v) \right) B(|v - v_*|, \theta) d\sigma dv_* . \]

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\( b \) is a scalar function depending on the cross section \( B \) only. If \( B(r, \theta) \approx r^{\gamma} \theta^{-d+1-2s} \) then \( b(v) \approx |v|^{\gamma} \).
The Boltzmann collision operator again

\[ Q(f, f)(v) = \int_{\mathbb{R}^n} \left( f(v') - f(v) \right) K_f(v, v') \, dv' + f(v)(b \ast f). \]

\( K_f \) is a kernel depending on \( f \) and the cross section \( B \). It can be computed after a somewhat difficult change of variables in the integral.

\( b \) is a scalar function depending on the cross section \( B \) only. If \( B(r, \theta) \approx r^\gamma \theta^{-d+1-2s} \) then \( b(v) \approx |v|^\gamma \).
The two terms

The first term

\[ \int_{\mathbb{R}^n} \left( f(v') - f(v) \right) K_f(v, v') \, dv' \]

is an integro-differential operator. It has exactly the form of the operators discussed before. It will have a regularization effect.

The second term

\[ f(v)(b \ast f) \]

This term doesn’t help. Here \( b(v) = |v|^\gamma \).

If \( \gamma \in [0, 2] \), this term is harmless because of the conservation of mass and second moment. The term is harder to control the more negative that \( \gamma \) is.
The diffusion kernel

$$K_f(v, v') = \frac{2^{2d-1}}{|v' - v|} \int_{\{w : w \cdot (v' - v) = 0\}} f(v + w) B(r, \theta) r^{-d+2} \, dw.$$ 

If $B(r, \theta) \approx r^\gamma \theta^{n+1-2s}$, then

$$K_f(v, v') \approx \left( \int_{\{w : w \cdot (v' - v) = 0\}} f(v + w) |w|^\gamma^{+2s+1} \, dw \right) |v' - v|^{-d-2s}.$$
Estimates for the kernel $K_f$

We are interested in estimates for $K_f$ depending only on the mass, energy and entropy of $f$.

The following holds

**Symmetry in the non-divergence way**

$$K(v, v + w) = K(v, v - w).$$

**Lower bound on a cone of directions**

$$K(v, v') \geq \lambda |v' - v|^{-d-2s},$$

for a set of values of $v'$ with positive measure.

**Upper bound on average**

$$\int_{B_R(v)} K_f(v, v') |v - v'|^2 \, dv' \leq \Lambda R^{2-2s}.$$
Bound below in a substantial set

Based only on the values of $M_0$, $M_1$, $E_0$ and $H_0$, we can prove that there is a $\delta > 0$, $R > 0$ and $\mu > 0$ so that for all $(t, x)$,

$$\left|\{v \in B_R : f(t, x, v) > \delta\}\right| \geq \mu.$$

This is used to obtain a $\lambda > 0$ so that

$$K_f(t, x, w) \geq \lambda |w|^{-(d-2s)},$$

if $w$ belongs to a thick cone of directions $A_{t, x}$.

(The thinkness of the cone $A_{t, x}$ shrinks as $|v| \to \infty$. The value of $\lambda$ actually grows.)
Bound below in a substantial set

We can prove $|\{v \in B_R : f(t, x, v) > \delta\}| \geq \mu$. Then

$K_f \gtrsim |v - v'|^{-d-2s}$ here

$f \geq \delta$ here
Proof of the Upper bound for $f$

Let

$$f(t, x_0, v_0) = \max_{x, v} f(t, x, v) = m(t).$$

At this point

$$m'(t) \leq f_t(t, x_0, v_0) = -v \cdot \nabla_x f + Q(f, f),$$

$$\nabla_x f(t, x_0, v_0) = 0,$$

$$Q(f, f) = \int_{\mathbb{R}^d} (f(v') - f(v_0)) K_f(v_0, v') \, dv' + (|v|^{\gamma} \ast f),$$

The first term is negative. The second term is positive.

(All is evaluated at $t$ and $x_0$)
Proof of the Upper bound for $f$

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$$Q(f, f) = \int_{\mathbb{R}^d} (f(v') - f(v_0)) K_f(v_0, v') \, dv' + (|v|^{\gamma} * f),$$

The first term is negative. The second term is positive.

(All is evaluated at $t$ and $x_0$)
Proof of the Upper bound for $f$

Let

$$f(t, x_0, v_0) = \max_{x, v} f(t, x, v) = m(t).$$

At this point

$$m'(t) \leq f_t(t, x_0, v_0) = -v \cdot \nabla_x f + Q(f, f),$$

$$\nabla_x f(t, x_0, v_0) = 0,$$

$$Q(f, f) = \int_{\mathbb{R}^d} (f(v') - m(t)) K_f(v_0, v') \, dv' + f(|v| \gamma * f),$$

The first term is negative. The second term is positive.

(All is evaluated at $t$ and $x_0$)
Which term wins?

\[ f (|v|^\gamma * f) \leq m(t)^{1+\gamma/d} M_0^{-\gamma/d} \quad \text{if } \gamma < 0. \]

\[ \int_{\mathbb{R}^d} (f(v') - m(t)) K_f(v_0, v') \, dv' < 0 \]

\[ K_f(v_0, v') \geq \lambda |v_0 - v'|^{-d-2s} \quad \text{when } v' \text{ is in a thick cone centered at } v_0. \]

Recall

\[ \int f(t, x, v') \, dv' \leq M_0. \]
Which term wins?

\[ f (|v|^{\gamma} \ast f) \leq m(t)^{1+\gamma/d} M_0^{-\gamma/d} \quad \text{if } \gamma < 0. \]

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Recall

\[ \int f(t, x, v') \, dv' \leq M_0. \]
The diffusion term

How negative is

$$\int_{\mathbb{R}^d} (f(v')-m(t))K_f(v_0, v') \, dv' < 0 \ ?$$

Given that $K_f(v_0, v') \geq \lambda |v_0 - v'|^{-d-2s}$ when $v'$ is in a thick cone centered at $v_0$, and $\int f(v') \, dv' \leq M_0$. 
The diffusion term

How negative is

$$\int_{\mathbb{R}^d} (f(v') - m(t)) K_f(v_0, v') \, dv' \leq -cm(t)^{1+2s/d}$$

Given that $K_f(v_0, v') \geq \lambda |v_0 - v'|^{-d-2s}$ when $v'$ is in a thick cone centered at $v_0$, and $\int f(v') \, dv' \leq M_0$. 
We obtain a contradiction when \( m(t) \) follows the following ODE.

\[
m'(t) = -cm(t)^{1+2s/d} + Cm(t)^{1-\gamma/d}.
\]

If \( 2s + \gamma > 0 \), this ODE gives us a finite upper bound \( m(t) \) (even with \( m(0) = +\infty \)).