Some Progress On Discrete Restriction

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(Joint work with Yi Hu)
Definitions

Let \( d \) and \( N \) be positive integers. Let \( f \) be a measurable function on \( \mathbb{T}^{d+1} \).
\( S_{d,N} \) is given by

\[
S_{d,N} = \{ (n_1, \cdots, n_d) \in \mathbb{Z}^d : |n_j| \leq N \text{ for all } 1 \leq j \leq d \}.
\]

\[
|n| = \sqrt{n_1^2 + \cdots + n_d^2}, \text{ for any } n = (n_1, \cdots, n_d) \in S_{d,N}
\]

Let \( A_{p,d,N} \) be the best constant satisfying

\[
\sum_{n \in S_{d,N}} \left| \hat{f}(n, |n|^2) \right|^2 \leq A_{p,d,N} \| f \|_{p'}^2
\]

(1)

Here \( p \) is a positive real number.
Bourgain’s conjecture:

\[
A_{p,d,N} \leq \begin{cases} 
C_p N^d - \frac{2(d+2)}{p} & \text{for } p > \frac{2(d+2)}{d} \\
C_p N^\varepsilon & \text{for } p = \frac{2(d+2)}{d} \\
C_p & \text{for } 2 \leq p < \frac{2(d+2)}{d}
\end{cases}
\]  \quad (2)
Bourgain’s Theorem

For $d = 1$,

$$A_{p,1,N} \leq \begin{cases} 
C_p N^{1 - \frac{6}{p}} & \text{for } p > 6 \\
C_p N^\varepsilon & \text{for } 4 < p \leq 6 \\
C_p & \text{for } 2 \leq p \leq 4 
\end{cases} \quad (3)$$

For $d = 2$,

$$A_{p,2,N} \leq \begin{cases} 
C_p N^{2 - \frac{8}{p}} & \text{for } p > 4 \\
C_p N^\varepsilon & \text{for } 2 < p \leq 4 \\
C_p & \text{for } p = 2 
\end{cases} \quad (4)$$
For $d = 3$,

$$A_{p,3,N} \leq C_p N^{3-\frac{10}{p}} \text{ for } p > 4$$

(5)

The critical index is $10/3$ if $d = 3$.

For $d \geq 4$

$$A_{p,d,N} \leq C_p N^{d-\frac{2(d+2)}{p}} \text{ for } p > \frac{2(d+4)}{d}$$

(6)

The critical index is $\frac{2(d+2)}{d}$ in this case.
The restriction estimate associated with $A_{p,d,N}$ implies

$$\left\| \sum_{n \in S_{d,N}} a_n e^{2\pi i (n \cdot x + |n|^2 t)} \right\|_{L^p(\mathbb{T}^{d+1})} \leq \sqrt{A_{p,d,N}} \left( \sum_n |a_n|^2 \right)^{1/2}.$$  \quad (7)

which is equivalent to the Strichartz type inequality

$$\left\| e^{it\Delta} \phi \right\|_{L^p(\mathbb{T}^{d+1})} \leq \sqrt{A_{p,d,N}} \| \phi \|_2$$  \quad (8)

for $\phi$ satisfying $\text{supp} \hat{\phi} \subset B(0, N)$.
By employing Bourgain’s results and performing the Picard iteration argument, one can obtain local (global) well-posedness for the following non-linear Schrödinger equation

$$\Delta_x u + i \partial_t u + u|u|^{\beta-2} = 0 \quad (\beta \geq 2) \quad (9)$$

with initial date

$$u(x,0) = \phi(x). \quad (10)$$

Here $\phi$ is periodic.
For instance,

**Theorem (Bourgain)**

*The Cauchy problem in 1D*

\[
\begin{align*}
\left\{ 
    u_{xx} + iu_t \pm u|u|^{p-2} &= 0 \\
    u(x,0) &= \phi(x)
\right.
\end{align*}
\]  

is locally well-posed for date \( \phi \in H^s(\mathbb{T}) \) when

\[
s > s_0, \quad p - 2 = \frac{4}{1 - 2s_0}, \quad p > 6
\]

or

\[
s > 0 \text{ and } 2 < p \leq 6.
\]
Questions related to Vinogradov mean value conjecture

**Question 1.** Let $d \geq 3$ be a positive integer. Suppose that $p > d(d + 1)$. Is it true that

$$
\sum_{n=1}^{N} \left| \hat{f}(n, n^2, \ldots, n^d) \right|^2 \leq C N^{1 - \frac{d(d+1)}{p}} \| f \|_p^2 \quad \text{? (12)}
$$

(12) is related to Strichartz's inequality and Vinogradov mean value conjecture.

**Vinogradov mean value conjecture**

Let $P(x, \alpha_1, \ldots, \alpha_d) = \sum_{j=1}^{d} \alpha_j x^j$ for $\alpha_1, \ldots, \alpha_d \in \mathbb{T}$. Let $J_d(N, b)$ be given by

$$
J_d(N, b) = \int_{\mathbb{T}^d} \left| \sum_{n=1}^{N} e^{2\pi i P(n, \alpha_1, \ldots, \alpha_d)} \right|^{2b} d\alpha_1 \ldots d\alpha_d.
$$

For positive integers $k$ and $b$, is it true that

$$
J_d(N, b) \leq C_{d,b,\varepsilon} \left( N^{b+\varepsilon} + N^{2b - \frac{d(d+1)}{2} + \varepsilon} \right) \quad \text{? (13)}
$$
**Question 2.** Let $d \geq 3$ be a positive integer. Suppose $p > 2(d + 1)$. Is it true that

$$
\sum_{n=1}^{N} \left| \hat{f}(n, n^d) \right|^2 \leq C N^{1 - \frac{2(d+1)}{p}} \| f \|_{p'}^2 \quad (14)
$$

When $d = 3$, (14) is related to the well-posedness problem for the periodic KDV equation:

$$
\partial_t u + \partial_x^3 u + u \partial_x u = 0
$$

with the initial periodic data

$$
u(x, 0) = \phi(x) .$$
Theorem (Bourgain)

\[ \sum_{n=1}^{N} \left| \hat{f}(n, n^3) \right|^2 \leq CN^\varepsilon \| f \|_6^2. \]

It seems that the following inequality is true, but we do not know how to prove it.

\[ \sum_{n=1}^{N} \left| \hat{f}(n, n^3) \right|^2 \leq CN^\varepsilon \| f \|_{8'}^2 \]  \hspace{1cm} (15)
Bourgain’s estimates on $A_{p,d,N}$ are based on

- Weyl’s sum estimates
- Hardy-Littlewood circle method
- Tomas-Stein restriction theorem
Large $p$ cases

For large $p$, Hardy-Littlewood circle method is enough for obtaining the desired estimates. Notice that

$$\sum_{n \in S_{d,N}} \left| \widehat{f}(n, |n|^2) \right|^2 \leq e^{\sigma d} \sum_{n \in S_{d,N}} e^{-\sigma |n|^2/N^2} \left| \widehat{f}(n, |n|^2) \right|^2 . \quad (16)$$

for any $\sigma > 0$.

Set

$$f_N(x, t) = \sum_{|n_j| \leq N, j=1,\ldots,d} \hat{f}(n, n_{d+1}) e^{2\pi i n \cdot x} e^{2\pi i n_{d+1} t},$$

which is the rectangular partial sum of Fourier series of $f$. Thus from (16),

$$\sum_{n \in S_{d,N}} \left| \widehat{f}(n, |n|^2) \right|^2 \leq e^{\sigma d} \sum_{n \in \mathbb{Z}^d} e^{-\sigma |n|^2/N^2} \left| \widehat{f}_N(n, |n|^2) \right|^2 . \quad (17)$$

for any $\sigma > 0$. 
Theorem (Hu and Li)

\[
\sum_{n \in \mathbb{Z}^d} e^{-\sigma|n|^2/N^2} \left| \hat{f}(n, |n|^2) \right|^2 \leq C_{\sigma,p} N^{d-\frac{2(d+2)}{p}} \| f \|^2_{p'},
\]

if \( p > [4 + 8/d] \).
Observe that
\[ \sum_{n \in \mathbb{Z}^d} e^{-\sigma |n|^2/N^2} \left| \hat{f}(n, |n|^2) \right|^2 \]
can be represented as
\[ \langle K_\sigma \ast f, f \rangle. \]

Here the kernel \( K_\sigma \) is a periodic function defined by
\[ K_\sigma(x, t) = \sum_{n \in \mathbb{Z}^d} e^{-\sigma |n|^2/N^2} e^{2\pi i n \cdot x} e^{2\pi i |n|^2 t}. \]
Dirichlet Principle. For any given $N \in \mathbb{N}$ and any $x \in (0, 1)$, there exist $a, q \in \mathbb{N}$ such that

$$\left| x - \frac{a}{q} \right| \leq \frac{1}{Nq},$$

$$1 \leq q \leq N, \quad a \in P_q.$$

Here

$$P_q = \{y \in \mathbb{N} : 1 \leq y \leq q, (y, q) = 1\}.$$

For $a \in P_q$, let

$$J_{a/q} = \left( -\frac{1}{Nq} + \frac{a}{q}, \frac{1}{Nq} + \frac{a}{q} \right).$$

If $q \leq N/100$, then $J_{a/q}$ is called a major arc. Let $\mathcal{M}$ be the union of all major arcs. Then

$$(0, 1) = \mathcal{M} \cup \mathcal{M}^c.$$
Decomposition of the kernel $K_\sigma$

$$K_\sigma(x, t) = K_\sigma(x, t) \sum_{J \in \mathcal{M}} 1_J + K_\sigma(x, t) \mathbf{1}_{\mathcal{M}^c}$$

$$:= K_{maj}(x, t) + K_{min}(x, t).$$

The main contribution for $p/2$ norm of the kernel is

$$\|K_{maj}\|_{L^{p/2}(\mathbb{T}^{d+1})} = \left( \sum_{q=1}^{N/100} \sum_{a \in \mathcal{P}_q} \|K_{a/q}\|_{p/2}^{p/2} \right)^{2/p}.$$

Here

$$K_{a/q}(x, t) = K_\sigma(x, t) \mathbf{1}_{J_{a/q}}.$$
A direct calculation yields

\[ \|K_{maj}\|_{L^{p/2}(\mathbb{T}^{d+1})} \leq CN^d \frac{2(d+1)}{p} \]

and therefore we obtain:

**Lemma**

*If* \( p > \lfloor 4 + 8/d \rfloor \), *then*

\[ \|K_\sigma\|_{p/2} \leq CN^d \frac{2(d+2)}{p} \].
Recall that
\[ \sum_{n \in \mathbb{Z}^d} e^{-\sigma|n|^2/N^2} \left| \hat{f}(n, |n|^2) \right|^2 = \langle K_\sigma * f, f \rangle. \]

The inner product is bounded by
\[ \| K_\sigma * f \|_p \| f \|_{p'} . \]

Applying Young’s inequality for the convolution $K_\sigma * f$, we can estimate the inner product by
\[ \| K_\sigma \|_{p/2} \| f \|_{p'}^2 . \]

Hence, we obtain our first theorem.
Recall that

\[ K_\sigma(x, t) = \sum_{n \in \mathbb{Z}^d} e^{-\sigma|n|^2/N^2} e^{2\pi i n \cdot x} e^{2\pi i |n|^2 t}. \]

For any \( Q \geq N \), this kernel can be decomposed into

\[ K_{1,Q}(x, t) + K_{2,Q}(x, t), \]

where \( K_{1,Q}, K_{2,Q} \) satisfies

\[ \| K_{1,Q} \|_\infty \leq CQ^{d+1/2}, \]

\[ \| \hat{K}_{2,Q} \|_{L^\infty(\mathbb{Z}^{d+1})} \leq \frac{CQ^\varepsilon}{Q}. \]
For any $\lambda > 0$, suppose that $\sum |a_n|^2 = 1$ and let $E_\lambda$ be

$$\left\{ (x, t) \in \mathbb{T}^d \times \mathbb{T} : \left| \sum_{n \in S_{d,N}} a_n e^{2\pi i (n \cdot x + |n|^2 t)} \right| > \lambda \right\}.$$ 

Then

$$\lambda^2 |E_\lambda|^2 \leq |\langle K_{\sigma} * f, f \rangle| \leq |\langle K_1, Q * f, f \rangle| + |\langle K_2, Q * f, f \rangle|.$$ 

Here $f \sim 1_{E_\lambda}$. 
Applying the $L^\infty$ estimates of $K_{1,Q}$ and $K_{2,Q}$, we have

**Theorem (Hu and Li)**

For $\lambda > 0$ and any $Q \geq N$,

$$\lambda^2 |E_\lambda|^2 \leq CQ^{\frac{d}{2}}|E_\lambda|^2 + \frac{CQ^\epsilon}{Q}|E_\lambda|.$$

For $\lambda > N^{\frac{d}{4}}$, set $Q \sim \lambda^{4/d}$. We then obtain

**Theorem (Bourgain)**

For $\lambda > N^{d/4}$,

$$|E_\lambda| \leq \frac{CN^\epsilon}{\lambda^{\frac{2(d+2)}{d}}}.$$
Discrete restriction associated with \((n, n^3)\)

**Theorem (Hu and Li)**

\[
\sum_{n=1}^{N} \left| \hat{f}(n, n^3) \right|^2 \leq CN^{1-\frac{8}{p}+\varepsilon} \| f \|_{p'}^2
\]  

(19)

for \( p \geq 14 \).

Here the following **Weyl sum** estimate is utilized

**Lemma (Weyl)**

*Suppose that* \( |t - a/q| \leq 1/q^2 \), *and that* \((a, q) = 1\). *Then*

\[
\left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + \alpha n^2 + \beta n)} \right| \leq C_\varepsilon N^{\frac{1}{4}+\varepsilon} q^{\frac{1}{4}}
\]

*if* \( q \geq N^2 \).
An application

**Theorem (L.K. Hua)**

\[
\int_{\mathbb{T} \times \mathbb{T}} \left| \sum_{n=1}^{N} e^{2\pi i (tn^3 + x n)} \right|^{10} \mathrm{d}x \mathrm{d}t \leq CN^{6+\varepsilon}.
\]  

(20)

**Corollary (L. K. Hua)**

\[
\int_{\mathbb{T}^3} \left| \sum_{n=1}^{N} e^{2\pi i (\alpha_3 n^3 + \alpha_2 n^2 + \alpha_3 n)} \right|^p \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \leq CN^{p-6+\varepsilon}
\]

for \( p > 16. \)
Corollary

\[ \sum_{n=1}^{N} \left| \hat{f}(n, n^2, n^3) \right|^2 \leq \frac{CN^{1-\frac{12}{p}+\epsilon}}{p'}, \quad (21) \]

if \( p > 32 \).

Conjecture. Is (21) true for \( p > 12 \) ?
Let

\[ F_N(x, t) = \sum_{n \in S_{d,N}} a_n e^{2\pi i (n \cdot x + |n|^2 t)} . \]

\( K_{p,d,N} \) denotes the best constant \( C_p \) satisfying

\[ \| F_N \|_p \leq C_p \| \{ a_n \} \|_{\ell^2} = C_p \left( \sum_n |a_n|^2 \right)^{1/2} \quad (22) \]

for all \( \{ a_n \} \in \ell^2 \).

It is easy to verify that \( K_{p,d,N} \sim \sqrt{A_{p,d,N}} \).
Estimates for $K_{p,d,N}$

Theorem (Hu and Li)

If $p > 0$ is an even integer, then $K_{p,d,N}$ is estimated by

$$
\sup_{l \in S_{d,pN/2}, m \in \{1, \ldots, pN^2/2\}} e^{2\pi \varepsilon m} \mathcal{F}_{\mathbb{T}^d \times \mathbb{T}}(F^{p/2}(\cdot, \cdot + i\varepsilon))(l, m).
$$

Here $\mathcal{F}_{\mathbb{T}^d \times \mathbb{T}}$ is Fourier transform of functions on $\mathbb{T}^d \times \mathbb{T}$, $\varepsilon$ is any real number, and $F$ is given by

$$
F(x, z) = \sum_{n \in \mathbb{Z}^d} e^{2\pi i z |n|^2 + 2\pi i x \cdot n}.
$$

(23)
Lemma (Bourgain)

Let $Q$ be a positive dyadic number and $M$ be a non-zero real number.

$$\sum_{q \sim Q} \left| \sum_{a \in \mathbb{P}_q} e^{2\pi i Ma/q} \right| \leq C \epsilon Q^{1+\epsilon} d(M; Q)$$

Here $d(M; Q)$ denotes the number of divisors of $M$ which are less than $Q$. 
THANK YOU !!!