Symmetry Reduction and Polyhedral Dynamics

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Motivation

Physical motivation:
- Study the quantization of gravity
- Investigate the proposal that space is quantized
- Discover the volume spectrum of tetrahedral grains of space

Mathematical Motivation:
- Provides convex polyhedra with a dynamical structure
- Pose and answer questions about polyhedra
- Leads to interesting integrable systems
Minkowski’s theorem: a tetrahedron

The area vectors of tetrahedron determine its shape:

$$
\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0.
$$
Physical input: impose $\vec{A}_1, \ldots, \vec{A}_4$ are angular momenta

Angular momenta have Poisson brackets,

$$\{f, g\} = \sum_{\ell=1}^{4} \vec{A}_\ell \cdot \left( \frac{\partial f}{\partial \vec{A}_\ell} \times \frac{\partial g}{\partial \vec{A}_\ell} \right)$$
$\vec{A}_1, \ldots, \vec{A}_4$ angular momenta

\[
p = |\vec{A}_1 + \vec{A}_2|, \quad q = \text{angle of rotation generated by } p:
\]

\[
\{q, p\} = 1
\]
Take as Hamiltonian the volume squared:

\[ H = V^2 = \frac{2}{9} \vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3). \]
$V_{\text{Tet}} = \frac{\sqrt{2}}{3} \times \frac{1}{\sqrt{\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)}}$

$A_1 = j + 1/2$
$A_2 = j + 1/2$
$A_3 = j + 1/2$
$A_4 = j + 3/2$

$\circ = \text{Numerical}$
$\bullet = \text{Bohr-Som}$

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Instead of a symplectic reduction, consider Poisson reduction by overall rotation (diagonal action), $\vec{A}_1 + \cdots + \vec{A}_4 (= 0)$.

Independent invariants: $\vec{A}_1 \cdot \vec{A}_2$, $\vec{A}_2 \cdot \vec{A}_3$, and $\vec{A}_1 \cdot \vec{A}_3$

Convenient coordinates: trade $\vec{A}_1 \cdot \vec{A}_2$ for $A \equiv |\vec{A}_1 + \vec{A}_2|^2$ and similarly, $B \equiv |\vec{A}_2 + \vec{A}_3|^2$ and $C \equiv |\vec{A}_1 + \vec{A}_3|^2$
Equations of motion

In these coordinates

\[ V^2 = ABC + aA + bB + cC + d, \]

where \( a, b, c, \) and \( d \) are constants (functions of \(|\vec{A}_1|, \ldots, |\vec{A}_4|\)).

Computation with the Lie-Poisson brackets gives,

\[ \dot{A} = \{A, V^2\} = A(B - C) + c - b, \]
\[ \dot{B} = \{B, V^2\} = B(C - A) + a - c, \]
\[ \dot{C} = \{C, V^2\} = C(A - B) + b - a, \]

a non-trivial deformation of the Lotka-Volterra system!
Evidently, $K \equiv A + B + C = \text{const.}$

An easy check demonstrates,

$$\dot{A} = \{A, K, V^2\} \equiv \left| \frac{\partial(A, K, V^2)}{\partial(A, B, C)} \right|, \cdots$$

This leads to two compatible Poisson structures,

$$\{f, g\}_K = -\vec{\nabla} K \cdot (\vec{\nabla} f \times \vec{\nabla} g) \quad \text{and} \quad \{f, g\}_{V^2} = \vec{\nabla} V^2 \cdot (\vec{\nabla} f \times \vec{\nabla} g)$$

i.e. this is a bi-Hamiltonian system.
Questions

This system does not fall into the usual classification of the Lotka-Volterra systems.

- Where does it belong in the bi-Hamiltonian classification?
- Is there a Lax pair for this system?
- What about an R-matrix and q-deformation?