Crash Course on Lie groupoid theory

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- **source** and **target** maps:

\[ \begin{array}{ccc}
  g & \leftrightarrow \ & M \\
  t(g) & \leftarrow \ & s(g)
\end{array} \]

- **product**:

\[ G^{(2)} = \{ (h, g) \in G \times G : s(h) = t(g) \} \]

\[ m : G^{(2)} \to G \]
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\[ \begin{align*}
    t(g) & \quad \bullet \quad s(g) \\
    \mathcal{G} & \quad \begin{array}{c}
        \xrightarrow{t} \\
        \xleftarrow{s}
    \end{array} \\
    & \quad M
\end{align*} \]

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    \end{array} \\
    g & \quad \begin{array}{c}
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- **Identity:**
  \[ u : M \rightarrow G \]

- **Inverse:**
  \[ \iota : G \rightarrow G \]

\[ g \quad g^{-1} \]

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This means we have a map $\mathcal{F} : \mathcal{G} \to \mathcal{H}$ between the sets of arrows, and a map $f : M \to N$ between the sets of objects, such that:

- if $g : x \to y$ is in $\mathcal{G}$, then $\mathcal{F}(g) : f(x) \to f(y)$ in $\mathcal{H}$.
- if $g, h \in \mathcal{G}$ are composable, then $\mathcal{F}(gh) = \mathcal{F}(g)\mathcal{F}(h)$.
- if $x \in M$, then $\mathcal{F}(1_x) = 1_{f(x)}$.
- if $g : x \to y$, then $\mathcal{F}(g^{-1}) = \mathcal{F}(g)^{-1}$. 

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**Crash Course on Lie groupoid theory**
Groupoids: basic concepts

- **right multiplication by** $g : y \leftarrow x$ is a bijection between $s$-fiber:
  \[ R_g : s^{-1}(y) \rightarrow s^{-1}(x), \quad h \mapsto hg. \]

- **left multiplication by** $g : y \leftarrow x$ is a bijection between $t$-fibers:
  \[ L_g : t^{-1}(x) \rightarrow t^{-1}(y), \quad h \mapsto gh. \]

- the **isotropy group at** $x$:
  \[ G_x = s^{-1}(x) \cap t^{-1}(x). \]

- the **orbit through** $x$:
  \[ O_x := t(s^{-1}(x)) = \{ y \in M : \exists g : x \rightarrow y \} \]
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**Definition**

A **Lie groupoid** is a groupoid $\mathcal{G} \Rightarrow M$ whose spaces of arrows and objects are both manifolds, the structure maps $s, t, u, m, i$ are all smooth maps and such that $s$ and $t$ are submersions.

**Basic Properties** For a Lie groupoid $\mathcal{G} \Rightarrow M$ and $x \in M$, one has that:

1. The isotropy groups $\mathcal{G}_x$ are Lie groups;
2. The orbits $O_x$ are (regular immersed) submanifolds in $M$;
3. The unit map $u : M \to \mathcal{G}$ is an embedding;
4. $t : s^{-1}(x) \to O_x$ is a principal $\mathcal{G}_x$-bundle.
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A picture of a Lie groupoid

\[ t-fibers \]

\[ s-fibers \]

\[ t(h) \quad s(h)=t(g) \quad s(g) \]

\[ G \quad M \]

\[ s(g) \quad t(g) \]

\[ h, g, hg \]
A picture of a Lie groupoid
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Lie groupoids vs (infinite dimensional) Lie groups

Definition

A **bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.
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![Diagram of Lie groupoids and bisections](image_url)
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- The group of bissections \( \Gamma(\mathcal{G}) \) is a Lie group (usually, infinite dimensional):
  - If \( \mathcal{G} = G \rightrightarrows \{\ast\} \), then \( \Gamma(\mathcal{G}) = G \);
  - If \( \mathcal{G} = M \times M \rightrightarrows M \), then \( \Gamma(\mathcal{G}) = \text{Diff}(M) \);
Lie groupoids vs (infinite dimensional) Lie groups

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A bisection of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $b : M \to \mathcal{G}$ such that $s \circ b : M \to M$ and $t \circ b : M \to M$ are diffeomorphisms.

- The group of bissections $\Gamma(\mathcal{G})$ is a Lie group (usually, infinite dimensional):

- One can use bissections to defined the groupoid of jets $J^k\mathcal{G} \rightrightarrows M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$. 
Some classes of groupoids

A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called:

- **source $k$-connected** if the $s$-fibers $s^{-1}(x)$ are $k$-connected for every $x \in M$. When $k = 0$ we say that $\mathcal{G}$ is a **s-connected groupoid**, and when $k = 1$ we say that $\mathcal{G}$ is a **s-simply connected groupoid**.

- **étale** if its source map $s$ is a local diffeomorphism.

- **proper** if the map $(s, t) : \mathcal{G} \to M \times M$ is a proper map.

**Remark.** Proper groupoids are the analogue of compact groups in Lie groupoid theory.
From Lie groupoids to Lie algebroids

\[ s(h) = t(g) \]
\[ s(g) \]
\[ t(h) \]
\[ h \]
\[ g \]
\[ t\text{-fibers} \]
\[ s\text{-fibers} \]

\[ t(h)g = s(h)t(g) \]
From Lie groupoids to Lie algebroids

A = Ker d s |_M
From Lie groupoids to Lie algebroids

\[ A = \text{Ker } d_s \bigg|_M \]

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From Lie groupoids to Lie algebroids

\[ A = \ker d s \mid_M \]

\[ R_g \]

\[ s(h) = t(g) \]

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\[ t\text{-fibers} \]

\[ G \]

\[ M \]

\[ h g \]

\[ h \]

\[ g \]

\[ \rho \]

\[ X, X \]

\[ \beta \]

\[ \alpha, \beta \]

\[ \rho \]

\[ R \]
From Lie groupoids to Lie algebroids

A = \ker d s \big|_M

[\alpha, \beta] = [X^\alpha, X^\beta]
From Lie groupoids to Lie algebroids

\[ A = \text{Ker } d\, s \bigg|_M \quad \rho = dt \bigg|_A \quad [\alpha, \beta] = [X^\alpha, X^\beta] \]
A Lie algebroid is a vector bundle $A \to M$ with:

(i) a Lie bracket $[\cdot,\cdot]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$;

(ii) a bundle map $\rho : A \to TM$ (the anchor);

such that:

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

- $\text{Im} \rho \subset TM$ is integrable $\Rightarrow$ characteristic foliation of $M$;
- For each $x \in M$, $\text{Ker} \rho_x$ is a finite dim Lie algebra (isotropy Lie algebra).
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The space of sections $\Gamma(A)$ is a Lie algebra (usually infinite dimensional):

- If $A = g \rightarrow \{\ast\}$, then $\Gamma(A) = g$;
- If $A = TM$, then $\Gamma(A) = \mathfrak{X}(M)$;

**Rmk.** If $G \rightrightarrows M$ is a Lie groupoid with Lie algebroid $A \rightarrow M$ one can define the **exponential map** $\exp : \Gamma(A) \rightarrow \Gamma(G)$. 
From Lie algebroids to Lie groupoids

**Theorem (Lie I)**

Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $A$. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\tilde{\mathcal{G}}$ with Lie algebroid $A$.

**Theorem (Lie II)**

Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with Lie algebroids $A$ and $B$, where $\mathcal{G}$ is source 1-connected. Given a Lie algebroid homomorphism $\phi : A \to B$, there exists a unique Lie groupoid homomorphism $\Phi : \mathcal{G} \to \mathcal{H}$ with $(\Phi)_* = \phi$.

... but Lie III does not hold!
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A non-integrable Lie algebroid

- Fix $\omega \in \Omega^2(M)$, closed, and take the associated Lie algebroid $A = TM \oplus \mathbb{R}$.

**Theorem**

The Lie algebroid $A$ integrates to a Lie groupoid $\mathcal{G}$ iff the group of spherical periods of $\omega$:

$$N_x := \left\{ \int_\gamma \omega \mid \gamma \in \pi_2(M, x) \right\} \subset \mathbb{R}$$

is discrete.

**Example**

If $M = S^2 \times S^2$ and $\omega = dA \oplus \lambda dA$, then $N_x$ is discrete iff $\lambda \in \mathbb{Q}$. 
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Obstructions to integrability

**Theorem (Crainic & RLF, 2003)**

For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.

Each $N_x$ is the image of a monodromy map:

$$\partial : \pi_2(L, x) \to \widetilde{G(g_x)}$$

with $L$ the leaf through $x$ and $g_x := \text{Ker } \rho_x$ the isotropy Lie algebra.

**Corollary**

A Lie algebroid $A$ is integrable provided either of the following hold:

(i) All leaves have finite $\pi_2$;
(ii) The isotropy Lie algebras have trivial center.
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The Maurer-Cartan Form on a Lie Groupoid

- For a Lie group $G$ the Maurer-Cartan form is the right-invariant $g$-valued 1-form:

$$\omega_{\text{MC}}(\xi) = (d_g R_{g^{-1}})(\xi) \in g.$$ 

It satisfies the Maurer-Cartan equation:

$$d\omega_{\text{MC}} + \frac{1}{2}[\omega_{\text{MC}}, \omega_{\text{MC}}] = 0.$$ 

- For a Lie groupoid $\mathcal{G}$, right translation by $g$ is a diffeomorphism 

$$R_g : s^{-1}(t(g)) \to s^{-1}(s(g)),$$

so the Maurer-Cartan form is a $s$-foliated 1-form.
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The Maurer-Cartan Form on a Lie Groupoid

**Definition**

The **Maurer-Cartan form on** $\mathcal{G}$ **is the s-foliated** $A$-valued 1-form defined by

$$\omega_{MC}(\xi) = (d_g R_{g^{-1}})(\xi) \in A_{t(g)}, \quad \xi \in T^s_{g}\mathcal{G}$$

The Maurer-Cartan form satisfies the Maurer-Cartan equation:

$$d\nabla \omega_{MC} + \frac{1}{2} [\omega_{MC}, \omega_{MC}]_{\nabla} = 0.$$
The Maurer-Cartan Form on on a Lie Groupoid

**Definition**

The **Maurer-Cartan form on** \( G \) **is the** \( s \)-**foliated** \( A \)-**valued 1-form**

\[
\omega_{MC}(\xi) = (dgR_{g^{-1}})(\xi) \in A_{t(g)}, \quad \xi \in T^s_{g}G
\]

- The Maurer-Cartan form satisfies the Maurer-Cartan equation:

\[
d\nabla \omega_{MC} + \frac{1}{2}[\omega_{MC}, \omega_{MC}]_{\nabla} = 0.
\]

for an auxiliary connection \( \nabla \).
Overlook

There is a huge body of results on Lie groupoid theory, developed in the last 15 years, which include, e.g.,:

- Normal form results and slice theorems for proper Lie groupoids.
- Van Est type theorems for Lie groupoid cohomology and vanishing theorems for cohomology of proper Lie groupoids.
- Equivariant cohomology and classifying spaces for Lie groupoids.
- Deformation cohomology, stability and rigidity results for Lie algebroids.
- Stratification structure for the orbit space of a proper Lie groupoid.

[...]
Choose local coordinates \((x^1, \ldots, x^d)\) on \(X\) and local basis of sections \(\{e_1, \ldots, e_n\}\) of \(A \to M\). Then:

\[
[e_i, e_j] = C^k_{ij}(x)e_k, \quad \rho(e_i) = B^a_i \frac{\partial}{\partial x^a}.
\]

Then:

- The Jacobi identity for \([\cdot, \cdot]_A\) gives:

\[
B^a_j \frac{\partial C^i_{k,l}}{\partial x^a} + B^a_k \frac{\partial C^i_{l,j}}{\partial x^a} + B^a_i \frac{\partial C^i_{j,k}}{\partial x^a} = (C^i_{m,j} C^m_{k,l} + C^i_{m,k} C^m_{l,j} + C^i_{m,l} C^m_{j,k})
\]

- The fact that \(\rho : \Gamma(A) \to \mathfrak{X}(M)\) preserves Lie brackets gives:

\[
B^b_i \frac{\partial B^a_j}{\partial x^b} - B^b_j \frac{\partial B^a_i}{\partial x^b} = C^l_{i,j} B^a_l.
\]