Happy Birthday eh!

Minneapolis May 17-20, 2012

Mark Fels
Utah State University
June 2011
Part 1

Symmetry, Congruence and Reconstruction
$\Delta = 0$ a system of differential equations.

$S$ the solution space.

$G$ a symmetry group of $\Delta = 0$ ($G$ acts on $S$)

I will think $\Delta = 0$ as

$\text{EDS } I \subset \Omega^* (\mathcal{M})$

Solutions:

$S = \{ s : \mathbb{N} \to \mathcal{M} \}$

Symmetry:

$\mu : G \times \mathcal{M} \to \mathcal{M}$,

$g \ast I = I \Rightarrow G$ acts on $S$

MOVING FRAMES: Can't think of a good reference...

Given $\mu : G \times \mathcal{M} \to \mathcal{M}$, submanifolds $s_1 : \mathbb{N} \to \mathcal{M}$ and $s_2 : \mathbb{N} \to \mathcal{M}$ are $G$-congruent if $s_2 = g \circ s_1$.

Are two solutions to $\Delta = 0$ $G$-congruent?
• $\Delta = 0$ a system of differential equations.
• $\Delta = 0$ a system of differential equations.
• $\mathcal{S}$ the solution space.
• $\Delta = 0$ a system of differential equations.
• $S$ the solution space.
• $G$ a symmetry group of $\Delta = 0$ ($G$ acts on $S$)
SYMMETRIES OF DIFFERENTIAL EQUATIONS: GTM Book

- \( \Delta = 0 \) a system of differential equations.
- \( S \) the solution space.
- \( G \) a symmetry group of \( \Delta = 0 \) (\( G \) acts on \( S \))

I will think \( \Delta = 0 \) as
• $\Delta = 0$ a system of differential equations.
• $\mathcal{S}$ the solution space.
• $G$ a symmetry group of $\Delta = 0$ ($G$ acts on $\mathcal{S}$)

I will think $\Delta = 0$ as

• EDS $\mathcal{I} \subset \Omega^*(M)$
SYMMETRIES OF DIFFERENTIAL EQUATIONS: GTM Book

- \( \Delta = 0 \) a system of differential equations.
- \( S \) the solution space.
- \( G \) a symmetry group of \( \Delta = 0 \) (\( G \) acts on \( S \))

I will think \( \Delta = 0 \) as

- EDS \( \mathcal{I} \subset \Omega^*(M) \)
- Solutions: \( S = \{ s : N \to M, \ s^*\mathcal{I} = 0 \} \)
• $\Delta = 0$ a system of differential equations.
• $S$ the solution space.
• $G$ a symmetry group of $\Delta = 0$ ($G$ acts on $S$)

I will think $\Delta = 0$ as

• EDS $\mathcal{I} \subset \Omega^*(M)$
• Solutions: $S = \{s : N \rightarrow M, \ s^*\mathcal{I} = 0\}$
• Symmetry: $\mu : G \times M \rightarrow M, \ g^*\mathcal{I} = \mathcal{I} \implies G$ acts on $S$
• \( \Delta = 0 \) a system of differential equations.
• \( S \) the solution space.
• \( G \) a symmetry group of \( \Delta = 0 \) (\( G \) acts on \( S \))

I will think \( \Delta = 0 \) as

• EDS \( \mathcal{I} \subset \Omega^*(M) \)
• Solutions: \( S = \{s : N \to M, \ s^*\mathcal{I} = 0\} \)
• Symmetry: \( \mu : G \times M \to M, \ g^*\mathcal{I} = \mathcal{I} \implies G \) acts on \( S \)

MOVING FRAMES: Can’t think of a good reference...
• \( \Delta = 0 \) a system of differential equations.
• \( S \) the solution space.
• \( G \) a symmetry group of \( \Delta = 0 \) (\( G \) acts on \( S \))

I will think \( \Delta = 0 \) as

• EDS \( \mathcal{I} \subset \Omega^*(M) \)
• Solutions: \( S = \{ s : N \to M, \ s^*\mathcal{I} = 0 \} \)
• Symmetry: \( \mu : G \times M \to M, \ g^*\mathcal{I} = \mathcal{I} \implies G \) acts on \( S \)

MOVING FRAMES: Can’t think of a good reference...

Given \( \mu : G \times M \to M \), submanifolds \( s_1 : N \to M \) and \( s_2 : N \to M \) are \( G \)-congruent if

\[
 s_2 = g \circ s_1.
\]
SYMMETRIES OF DIFFERENTIAL EQUATIONS: GTM Book

- \( \Delta = 0 \) a system of differential equations.
- \( S \) the solution space.
- \( G \) a symmetry group of \( \Delta = 0 \) (\( G \) acts on \( S \))

I will think \( \Delta = 0 \) as

- EDS \( \mathcal{I} \subset \Omega^*(M) \)
- Solutions: \( S = \{ s : N \to M, \ s^*\mathcal{I} = 0 \} \)
- Symmetry: \( \mu : G \times M \to M, \ g^*\mathcal{I} = \mathcal{I} \implies G \) acts on \( S \)

MOVING FRAMES: Can't think of a good reference...

Given \( \mu : G \times M \to M \), submanifolds \( s_1 : N \to M \) and \( s_2 : N \to M \) are \( G \)-congruent if

\[
  s_2 = g \circ s_1. 
\]

Are two solutions to \( \Delta = 0 \) \( G \)-congruent?
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.

Start with a simple example.
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.

Start with a simple example.

Let $G = \{(a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$ act on $J^2(\mathbb{R}, \mathbb{R}) - \{v_x = 0\}$ by

$$(a, b) \cdot (x, v, v_x, v_{xx}) = (x, av + b, av_x, a v_{xx})$$
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.

Start with a simple example.

Let $G = \{(a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$ act on $J^2(\mathbb{R}, \mathbb{R}) - \{v_x = 0\}$ by

$$(a, b) \cdot (x, v, v_x, v_{xx}) = (x, av + b, av_x, a v_{xx})$$

The infinitesimal generators are

$$X_a = v \partial_v + v_x \partial_{v_x} + v_{xx} \partial_{v_{xx}}, \quad X_b = \partial_v.$$
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.

Start with a simple example.

Let $G = \{(a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$ act on $J^2(\mathbb{R}, \mathbb{R}) - \{v_x = 0\}$ by

$$(a, b) \cdot (x, v, v_x, v_{xx}) = (x, av + b, av_x, a v_{xx})$$

The infinitesimal generators are

$$X_a = v \partial_v + v_x \partial_{v_x} + v_{xx} \partial_{v_{xx}}, \quad X_b = \partial_v.$$  

The invariants are

$$x, \quad \frac{v_{xx}}{v_x}$$
At the end of this part, I’ll give necessary and sufficient conditions that two solutions $s_1, s_2$ to a system of differential equations are $G$-congruent.

Start with a simple example.

Let $G = \{(a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R}\}$ act on $J^2(\mathbb{R}, \mathbb{R}) - \{v_x = 0\}$ by

$$(a, b) \cdot (x, v, v_x, v_{xx}) = (x, av + b, av_x, a v_{xx})$$

The infinitesimal generators are

$$X_a = v \partial_v + v_x \partial_{v_x} + v_{xx} \partial_{v_{xx}}, \quad X_b = \partial_v.$$ 

The invariants are

$$x, \quad \frac{v_{xx}}{v_x}$$

The quotient $q : J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})/G$ is given in terms of the invariants as

$$q(x, v, v_x, v_{xx}) = \left(x, \hat{\xi} = \frac{v_{xx}}{v_x}\right)$$
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]

So what about the "prescribed curvature problem"?
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]

So what about the "prescribed curvature problem"?

Given \( \hat{\xi} = G(x) \) how do we find a curve with given curvature \( G(x) \)?
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]

So what about the "prescribed curvature problem"?

Given \( \hat{\xi} = G(x) \) how do we find a curve with given curvature \( G(x) \)?

I will call this the Reconstruction problem.
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]

So what about the "prescribed curvature problem"?

Given \( \hat{\xi} = G(x) \) how do we find a curve with given curvature \( G(x) \)?

I will call this the **Reconstruction** problem.

The curve will be unique up to congruence.
The method of moving frames in this example tells us that

\[ s_1(x) = g \cdot s_2(x) \]

if and only if

\[ \hat{\xi}_2 = \frac{v_{xx}}{v_x} = \frac{(s_2)''}{(s_2)'} = \frac{(s_1)''}{(s_1)'} = \hat{\xi}_1(x). \]

So what about the "prescribed curvature problem"?

Given \( \hat{\xi} = G(x) \) how do we find a curve with given curvature \( G(x) \)?

I will call this the Reconstruction problem.

The curve will be unique up to congruence.

Look at three different methods for finding the such a curve.
Method 1

Given $\dot{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \dot{\xi})$ space

$$\bar{s}(x) = (x, \dot{\xi} = G(x)).$$
Method 1

Given \( \hat{\xi} = \frac{v_{xx}}{v_x} = G(x) \), it is a curve in \((x, \hat{\xi})\) space

\[ \bar{s}(x) = (x, \hat{\xi} = G(x)). \]

1. choose a curve \( \hat{s} : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R}) \) such that \( q \circ \hat{s} = \bar{s} \). For example,

\[ \hat{s}(x) = (x, v = 0, v_x = 1, v_{xx} = G(x)) \]
Method 1

Given $\hat{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. Choose a curve $\hat{s}: \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, v = 0, v_x = 1, v_{xx} = G(x))$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0: \mathbb{R} \to \mathbb{R}$
Method 1

Given $\hat{\xi} = \frac{\nu_{xx}}{\nu_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. choose a curve $\hat{s} : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, \nu = 0, \nu_x = 1, \nu_{xx} = G(x)).$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0 : \mathbb{R} \to \mathbb{R}$.

If so the contact system $d\nu - \nu_x dx, d\nu_x - \nu_{xx} dx$ would vanish on $\hat{s}$, but

$$\hat{s}^*(d\nu - \nu_x dx) \neq 0, \quad \hat{s}^*(d\nu_x - \nu_{xx} dx) \neq 0.$$
Method 1

Given $\hat{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. Choose a curve $\hat{s} : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, v = 0, v_x = 1, v_{xx} = G(x))$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0 : \mathbb{R} \to \mathbb{R}$.

If so the contact system $dv - v_x dx$, $dv_x - v_{xx} dx$ would vanish on $\hat{s}$, but

$$\hat{s}^*(dv - v_x dx) \neq 0, \quad \hat{s}^*(dv_x - v_{xx} dx) \neq 0.$$

2. Perturb $\hat{s}$ by a curve $\gamma : \mathbb{R} \to G$.  

Method 1

Given $\hat{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. choose a curve $\hat{s} : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, \nu = 0, \nu_x = 1, \nu_{xx} = G(x))$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0 : \mathbb{R} \to \mathbb{R}$

If so the contact system $dv - v_x dx, dv_x - v_{xx} dx$ would vanish on $\hat{s}$, but

$$\hat{s}^*(dv - v_x dx) \neq 0, \quad \hat{s}^*(dv_x - v_{xx} dx) \neq 0.$$

2. Perturb $\hat{s}$ by a curve $\gamma : \mathbb{R} \to G$

$$s(x) = \mu(\gamma(x), \hat{s})$$

$$= (a(x), b(x)) \cdot (x, \nu = 0, \nu_x = 1, \nu_{xx} = G(x))$$

$$= (x, b(x), a(x), a(x)G(x))$$
Method 1

Given $\hat{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. choose a curve $\hat{s} : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, v = 0, v_x = 1, v_{xx} = G(x))$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0 : \mathbb{R} \to \mathbb{R}$

If so the contact system $dv - v_x dx, dv_x - v_{xx} dx$ would vanish on $\hat{s}$, but

$$\hat{s}^*(dv - v_x dx) \neq 0, \quad \hat{s}^*(dv_x - v_{xx} dx) \neq 0.$$

2. Perturb $\hat{s}$ by a curve $\gamma : \mathbb{R} \to G$

$$s(x) = \mu(\gamma(x), \hat{s})$$

$$= (a(x), b(x)) \cdot (x, v = 0, v_x = 1, v_{xx} = G(x))$$

$$= (x, b(x), a(x), a(x)G(x))$$

requiring $s$ satisfies the contact conditions (the curve is a prolongation).
Method 1

Given $\hat{\xi} = \frac{v_{xx}}{v_x} = G(x)$, it is a curve in $(x, \hat{\xi})$ space

$$\bar{s}(x) = (x, \hat{\xi} = G(x)).$$

1. **Choose a curve** $\hat{s} : \mathbb{R} \rightarrow J^2(\mathbb{R}, \mathbb{R})$ such that $q \circ \hat{s} = \bar{s}$. For example,

$$\hat{s}(x) = (x, v = 0, v_x = 1, v_{xx} = G(x))$$

Note that $\hat{s}$ is not the prolongation of a curve $s_0 : \mathbb{R} \rightarrow \mathbb{R}$.

If so the contact system $dv - v_x dx, dv_x - v_{xx} dx$ would vanish on $\hat{s}$, but

$$\hat{s}^*(dv - v_x dx) \neq 0, \quad \hat{s}^*(dv_x - v_{xx} dx) \neq 0.$$ 

2. **Perturb $\hat{s}$ by a curve** $\gamma : \mathbb{R} \rightarrow G$

$$s(x) = \mu(\gamma(x), \hat{s})$$

$$= (a(x), b(x)) \cdot (x, v = 0, v_x = 1, v_{xx} = G(x))$$

$$= (x, b(x), a(x), a(x)G(x))$$

requiring $s$ satisfies the contact conditions (the curve is a prolongation).

3. **Solve the resulting ODE for** $\gamma : \mathbb{R} \rightarrow G$. 

From above

\[ s(x) = \mu(\gamma(x), \hat{s}) = (x, \ v = b(x), \ v_x = a(x), \ v_{xx} = a(x)G(x) ) \]  \quad (1)
From above

\[ s(x) = \mu(\gamma(x), \hat{s}) = (x, \ v = b(x), \ v_x = a(x), \ v_{xx} = a(x)G(x)) \]  

(1)

the contact conditions are

\[ \frac{db}{dx} = a(x), \quad \frac{da}{dx} = a(x)G(x) \]  

(2)
From above

\[ s(x) = \mu(\gamma(x), \hat{s}) = (x, v = b(x), v_x = a(x), v_{xx} = a(x)G(x)) \]  

(1)

the contact conditions are

\[ \frac{db}{dx} = a(x), \quad \frac{da}{dx} = a(x)G(x) \]

(2)

If we include the initial condition \( a(0) = 1, b(0) = 0 \) then the solutions is

\[ a(x) = e^{\int_{\gamma}^{x} G(t) dt}, \quad b(x) = \int_{0}^{x} \left( e^{\int_{s}^{x} G(t) dt} \right) ds. \]
From above

\[ s(x) = \mu(\gamma(x), \hat{s}) = (x, v = b(x), v_x = a(x), v_{xx} = a(x)G(x)) \]  \hspace{1cm} (1) 

the contact conditions are

\[ \frac{db}{dx} = a(x), \hspace{0.5cm} \frac{da}{dx} = a(x)G(x) \]  \hspace{1cm} (2) 

If we include the initial condition \( a(0) = 1, b(0) = 0 \) then the solutions is

\[ a(x) = e^{\int_0^x G(t)dt}, \hspace{0.5cm} b(x) = \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds. \]

Giving the curve \( s(x) \) in (1) to be

\[ v(x) = b(x) = \int_0^x \left( e^{\int_0^t G(s)ds} \right) dt \]
From above

\[ s(x) = \mu(\gamma(x), \hat{s}) = (x, v = b(x), v_x = a(x), v_{xx} = a(x)G(x)) \]  

(1)

the contact conditions are

\[ \frac{db}{dx} = a(x), \quad \frac{da}{dx} = a(x)G(x) \]  

(2)

If we include the initial condition \( a(0) = 1, b(0) = 0 \) then the solutions is

\[ a(x) = e^{\int_0^x G(t)dt}, \quad b(x) = \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds. \]

Giving the curve \( s(x) \) in (1) to be

\[ v(x) = b(x) = \int_0^x \left( e^{\int_0^t G(s)ds} \right) dt \]

Remark: Equations (2) are equations of Lie type.
Method 2

Again we have prescribed

\[
\frac{v_{xx}}{v_x} = \hat{\xi} = G(x)
\]
Method 2

Again we have prescribed

\[
\frac{v_{xx}}{v_x} = \hat{\xi} = G(x)
\]

Consider \( P \subset J^2(\mathbb{R}, \mathbb{R}) \) where

\[
P = q^{-1}(\bar{s}(x)) = q^{-1}(x, \hat{\xi} = G(x)) = (x, v, v_x, v_{xx} = G(x)v_x)
\]
Method 2

Again we have prescribed
\[ \frac{v_{xx}}{v_x} = \hat{\xi} = G(x) \]

Consider \( P \subset J^2(\mathbf{R}, \mathbf{R}) \) where
\[ P = q^{-1}(\bar{s}(x)) = q^{-1}(x, \hat{\xi} = G(x)) = (x, v, v_x, v_{xx} = G(x)v_x) \]

On \( P \) the coordinates are \((x, v, v_x)\) and the contact system is
\[ I = \langle \theta = dv - v_x dx, \quad \theta_x = dv_x - G(x)v_x dx \rangle. \]
Method 2

Again we have prescribed
\[
\frac{v_{xx}}{v_x} = \hat{\xi} = G(x)
\]

Consider \( P \subset J^2(\mathbb{R}, \mathbb{R}) \) where
\[
P = q^{-1}(\bar{s}(x)) = q^{-1}(x, \hat{\xi} = G(x)) = (x, v, v_x, v_{xx} = G(x)v_x)
\]

On \( P \) the coordinates are \((x, v, v_x)\) and the contact system is
\[
\mathcal{I} = \langle \theta = dv - v_x dx, \quad \theta_x = dv_x - G(x)v_x dx \rangle.
\]

\( \mathcal{I} \) is completely integrable so find two independent first integrals \( f_i(x, v, v_x) \), for \( \mathcal{I} \),
\[
df_i \in \text{span}\{\theta, \theta_x\}.
\]
Method 2

Again we have prescribed
\[
\frac{v_{xx}}{v_x} = \hat{\xi} = G(x)
\]

Consider \( P \subset J^2(\mathbb{R}, \mathbb{R}) \) where
\[
P = q^{-1}(\bar{s}(x)) = q^{-1}(x, \hat{\xi} = G(x)) = (x, v, v_x, v_{xx} = G(x)v_x)
\]

On \( P \) the coordinates are \((x, v, v_x)\) and the contact system is
\[
\mathcal{I} = \langle \theta = dv - v_x dx, \theta_x = dv_x - G(x)v_x dx \rangle.
\]

\( \mathcal{I} \) is completely integrable so find two independent first integrals \( f_i(x, v, v_x) \), for \( \mathcal{I} \),
\[
df_i \in \text{span}\{\theta, \theta_x\}.
\]

The solutions \( s : \mathbb{R} \rightarrow P \) satisfying the contact conditions \( s^*\mathcal{I} = 0 \) lie on level sets
\[
f_i(x, v, v_x) = c_i.
\]
Method 2

Again we have prescribed

\[ \frac{v_{xx}}{v_x} = \xi = G(x) \]

Consider \( P \subset J^2(\mathbb{R}, \mathbb{R}) \) where

\[ P = q^{-1}(\bar{s}(x)) = q^{-1}(x, 0, 0, v_{xx} = G(x)v_x) = (x, v, v_x, v_{xx} = G(x)v_x) \]

On \( P \) the coordinates are \((x, v, v_x)\) and the contact system is

\[ \mathcal{I} = \langle \theta = dv - v_x dx, \quad \theta_x = dv_x - G(x)v_x dx \rangle. \]

\( \mathcal{I} \) is completely integrable so find two independent first integrals \( f_i(x, v, v_x) \), for \( \mathcal{I} \),

\[ df_i \in \text{span}\{\theta, \theta_x\}. \]

The solutions \( s : \mathbb{R} \to P \) satisfying the contact conditions \( s^*\mathcal{I} = 0 \) lie on level sets

\[ f_i(x, v, v_x) = c_i. \]

\( G \) is solvable so the first integrals can be found by quadratures.
Here is how: The infinitesimal generators of $av + b$ action on $P$ are

$$X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.$$
Here is how: The infinitesimal generators of $av + b$ action on $P$ are

$$X_a = v\partial_v + v_x\partial_{v_x}, \quad X_b = \partial_v.$$ 

Find $\omega^i \in \mathcal{I}$ satisfying $\omega^i(X_j) = \delta^i_j$.
Here is how: The infinitesimal generators of \( av + b \) action on \( P \) are
\[
X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.
\]

Find \( \omega^i \in \mathcal{I} \) satisfying \( \omega^i(X_j) = \delta^i_j \)
\[
\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x) v_x dx)
\]
\[
\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x) v_x dx)
\]
Here is how: The infinitesimal generators of $av + b$ action on $P$ are

$$X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.$$  

Find $\omega^i \in \mathcal{I}$ satisfying $\omega^i(X_j) = \delta^i_j$

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

These satisfy

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0$$
Here is how: The infinitesimal generators of $a v + b$ action on $P$ are

$$X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.$$ 

Find $\omega^i \in I$ satisfying $\omega^i(X_j) = \delta^i_j$

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

These satisfy

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0$$

therefore

$$\omega_2 = df_2(x, v, v_x), \quad \omega^1 = e^{f_2} df_1$$
Here is how: The infinitesimal generators of $av + b$ action on $P$ are

$$X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.$$ 

Find $\omega^i \in \mathcal{I}$ satisfying $\omega^i(X_j) = \delta^i_j$

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = d\nu - v_x dx - \frac{\nu}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

These satisfy

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0$$

therefore

$$\omega_2 = df_2(x, \nu, v_x), \quad \omega^1 = e^{f_2} df_1$$

where (using DeRham homotopy for example),

$$f_2 = \log v_x - \int G(x) dx, \quad f_1 = \frac{\nu}{v_x} e^{\int G(x) dx} - \int e^{\int G(x) dx} dx.$$
Here is how: The infinitesimal generators of $av + b$ action on $P$ are

$$X_a = v \partial_v + v_x \partial_{v_x}, \quad X_b = \partial_v.$$ 

Find $\omega^i \in \mathcal{I}$ satisfying $\omega^i(X_j) = \delta^i_j$

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

These satisfy

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0$$

therefore

$$\omega_2 = df_2(x, v, v_x), \quad \omega^1 = e^{f_2} df_1$$

where (using DeRham homotopy for example),

$$f_2 = \log v_x - \int G(x) dx, \quad f_1 = \frac{v}{v_x} e^{\int G(x) dx} - \int e^{\int G(x) dx} dx.$$ 

With the initial conditions $f_1(0, 0, 1) = 0, f_2(0, 0, 1) = 0$ we have

$$f_2 = \log v_x - \int_0^x G(t) dt, \quad f_1 = \frac{v}{v_x} e^{\int_0^x G(t) dt} - \int_0^x e^{\int_0^s G(t) dt} ds.$$
With

\[ f_2 = \log v_x - \int_0^x G(t) dt, \quad f_1 = \frac{V}{v_x} e^{\int_0^x G(t) dt} - \int_0^x \left( e^{\int_s^0 G(t) dt} \right) ds \]
With

\[ f_2 = \log v_x - \int_0^x G(t)dt, \quad f_1 = \frac{V}{v_x} e^{\int_0^x G(t)dt} - \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds \]

the level set \( f_1 = 0, \ f_2 = 0 \) is

\[ v_x = e^{\int_0^x G(t)dt}, \quad v(x) = \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds \]
With

\[ f_2 = \log v_x - \int_0^x G(t) dt, \quad f_1 = \frac{v}{v_x} e^{\int_0^x G(t) dt} - \int_0^x \left( e^{\int_0^s G(t) dt} \right) ds \]

the level set \( f_1 = 0, \ f_2 = 0 \) is

\[ v_x = e^{\int_0^x G(t) dt}, \quad v(x) = \int_0^x \left( e^{\int_0^s G(t) dt} \right) ds \]

The same solution as Method 1.
With
\[ f_2 = \log v_x - \int_0^x G(t) \, dt, \quad f_1 = \frac{v}{v_x} e^{\int_0^x G(t) \, dt} - \int_0^x \left( e^{\int_0^s G(t) \, dt} \right) \, ds \]

the level set \( f_1 = 0, \ f_2 = 0 \) is
\[ v_x = e^{\int_0^x G(t) \, dt}, \quad v(x) = \int_0^x \left( e^{\int_0^s G(t) \, dt} \right) \, ds \]

The same solution as Method 1.

The group action is not needed with this method, only the infinitesimal generators and the DeRham homotopy formula.
Method 3

We have from Method 2 the differential forms on $P = q^{-1}(\bar{s}(x))$,

\[ \omega^1 = \theta - \frac{1}{\nu_x} \theta_x = d\nu - \nu_x dx - \frac{\nu}{\nu_x} (d\nu_x - G(x) \nu_x dx) \]

\[ \omega^2 = \frac{1}{\nu_x} \theta_x = \frac{1}{\nu_x} (d\nu_x - G(x) \nu_x dx) \]
Method 3

We have from Method 2 the differential forms on \( P = q^{-1}(\bar{s}(x)) \),

\[
\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)
\]

\[
\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)
\]

which satisfy,

\[
d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0.
\]
Method 3

We have from Method 2 the differential forms on $P = q^{-1}(\bar{s}(x))$,

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

which satisfy,

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0.$$ 

Find $\rho : P \to G$ with $\rho^* \tau^i = \omega^i$ where $\tau^i$ are the Maurer Cartan forms

$$\tau^a = \frac{da}{a}, \quad \tau^b = db - \frac{b}{a} da.$$
Method 3

We have from Method 2 the differential forms on $P = q^{-1}(\bar{s}(x))$,

$$\omega^1 = \theta - \frac{1}{v_x} \theta_x = dv - v_x dx - \frac{v}{v_x} (dv_x - G(x)v_x dx)$$

$$\omega^2 = \frac{1}{v_x} \theta_x = \frac{1}{v_x} (dv_x - G(x)v_x dx)$$

which satisfy,

$$d\omega^1 = -\omega^1 \wedge \omega^2, \quad d\omega^2 = 0.$$ 

Find $\rho : P \to G$ with $\rho^* \tau^i = \omega^i$ where $\tau^i$ are the Maurer Cartan forms

$$\tau^a = \frac{da}{a}, \quad \tau^b = db - \frac{b}{a} da.$$ 

The map $\rho$ can be found by quadratures (like finding the first integrals)

$$a = v_x e^{-\int_0^x G(t)dt}, \quad b = v - v_x e^{-\int_0^x G(t)dt} \int_0^x \left( e^{\int_s^x G(t)dt} \right) ds$$

where we have chosen $\rho(0,0,1) = (1,0) = e \in G$
With $\rho : P \to G$ given, any choice of a curve $\hat{s} : \mathbb{R} \to P$ gives

$$s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))$$

is a solution.
With $\rho : P \to G$ given, any choice of a curve $\hat{s} : \mathbb{R} \to P$ gives

$$s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))$$

is a solution.

From previous slide, $\rho(x, v, v_x) \to (a, b)$ is

$$a = v_x e^{-\int_0^x G(t)dt}, \quad b = v - v_x e^{-\int_0^x G(t)dt} \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds$$
With $\rho : P \to G$ given, any choice of a curve $\hat{s} : \mathbb{R} \to P$ gives

$$s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))$$

is a solution.

From previous slide, $\rho(x, v, v_x) \to (a, b)$ is

$$a = v_x e^{-\int_0^x G(t)dt}, \quad b = v - v_x e^{-\int_0^x G(t)dt} \int_0^x \left(e^{\int_0^s G(t)dt}\right) ds$$

then with $\hat{s}(x) = (x, v = 0, v_x = 1)$,

$$\rho(\hat{s}(x)) = \begin{pmatrix} a = e^{-\int_0^x G(t)dt}, b = -e^{-\int_0^x G(t)dt} \int_0^x \left(e^{\int_0^s G(t)dt}\right) ds \end{pmatrix}$$
With $\rho : P \to G$ given, any choice of a curve $\hat{s} : \mathbb{R} \to P$ gives

$$s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))$$

is a solution.

From previous slide, $\rho(x, v, v_x) \to (a, b)$ is

$$a = v_x e^{-\int_0^x G(t)dt}, \quad b = v - v_x e^{-\int_0^x G(t)dt} \int_0^x \left(e^{\int_0^s G(t)dt}\right) ds$$

then with $\hat{s}(x) = (x, v = 0, v_x = 1)$,

$$\rho(\hat{s}(x)) = \left(a = e^{-\int_0^x G(t)dt}, b = -e^{-\int_0^x G(t)dt} \int_0^x \left(e^{\int_0^s G(t)dt}\right) ds\right)$$

and

$$s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))$$

$$= \left(x, \quad v = -\frac{b}{a} = \int_0^x \left(e^{\int_0^s G(t)dt}\right) ds, \quad v_x = \frac{1}{a} = e^{\int_0^x G(t)dt}\right)$$
With \( \rho : P \to G \) given, any choice of a curve \( \hat{s} : \mathbb{R} \to P \) gives

\[
s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))
\]

is a solution.

From previous slide, \( \rho(x, v, v_x) \to (a, b) \) is

\[
a = v_x e^{-\int_0^x G(t)dt}, \quad b = v - v_x e^{-\int_0^x G(t)dt} \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds
\]

then with \( \hat{s}(x) = (x, v = 0, v_x = 1) \),

\[
\rho(\hat{s}(x)) = \left( a = e^{-\int_0^x G(t)dt}, b = -e^{-\int_0^x G(t)dt} \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds \right)
\]

and

\[
s(x) = \mu(\rho(\hat{s}(x))^{-1}, \hat{s}(x))
\]

\[
= \left( x, \quad v = -\frac{b}{a} = \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds, \quad v_x = \frac{1}{a} = e^{\int_0^x G(t)dt} \right)
\]

Same as before.
Remark: If instead we worked on $J^3(\mathbb{R}, \mathbb{R})$ with

$$X_c = v^2 \partial_v + 2vv_x \partial_{v_x} + 2(v_x^2 + vv_{xx}) \partial_{v_{xx}} + 2(3v_x v_{xx} + vv_{xxx}) \partial_{v_{xxx}}$$

(so $\mathfrak{sl}(2)$) then the invariant would be the Schwartzian,
**Remark:** If instead we worked on $J^3(\mathbb{R}, \mathbb{R})$ with

$$X_c = v^2 \partial_v + 2vv_x \partial_v + 2(v_x^2 + vv_{xx}) \partial_{v_{xx}} + 2(3v_x v_{xx} + vv_{xxx}) \partial_{v_{xxx}}$$

(so $\mathfrak{sl}(2)$) then the invariant would be the Schwartzian,

$$\hat{\xi} = \frac{v_{xxx}}{v_x} - \frac{3}{2} \left( \frac{v_{xx}}{v_x} \right)^2$$

(Or curves into $\mathbb{RP}^1$ with global $SL(2)$ action)
Remark: If instead we worked on $J^3(\mathbb{R}, \mathbb{R})$ with

\[ X_c = v^2 \partial_v + 2vv_x \partial_{v_x} + 2(v_x^2 + vv_{xx}) \partial_{v_{xx}} + 2(3v_xv_{xx} + vv_{xxx}) \partial_{v_{xxx}} \]

(so \( \mathfrak{sl}(2) \)) then the invariant would be the Schwartzian,

\[ \hat{\xi} = \frac{v_{xxx}}{v_x} - \frac{3}{2} \left( \frac{v_{xx}}{v_x} \right)^2 \]

(Or curves into \( \mathbb{RP}^1 \) with global \( SL(2) \) action)

The reconstruction problem - given \( \hat{\xi} = G(x) \) a curve exists and its determination is an equation of Lie type for \( SL(2) \). Generically these can’t be integrated by quadratures.
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$

**Congruence:** Two integral manifolds $s_1, s_2 : N \rightarrow M$ with $N$ connected, are congruent if and only if the projection $q \circ s_1 = q \circ s_2$ are the same solutions to the quotient

$$\bar{\mathcal{I}} = \{ \bar{\theta} \in \Omega^*(M/G) \mid q^*\bar{\theta} \in \mathcal{I}, \quad q : M \rightarrow M/G \}. $$
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$

**Congruence:** Two integral manifolds $s_1, s_2 : N \to M$ with $N$ connected, are congruent if and only if the projection $q \circ s_1 = q \circ s_2$ are the same solutions to the quotient

$$\tilde{\mathcal{I}} = \{ \tilde{\theta} \in \Omega^*(M/G) \mid q^* \tilde{\theta} \in \mathcal{I}, \quad q : M \to M/G \}.$$

**Reconstruction:** Given a solution $\tilde{s} : N \to M/G$ of $\mathcal{I}/G$, a (local) solution to $\mathcal{I}$ on $M$ can be found by solving a system of equations of Lie type (using one of the 3 methods outlined in the example). If the group $G$ is solvable, these can be integrated by quadratures.
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$ 

**Congruence:** Two integral manifolds $s_1, s_2 : N \to M$ with $N$ connected, are congruent if and only if the projection $q \circ s_1 = q \circ s_2$ are the same solutions to the quotient

$$\mathcal{I} = \{ \bar{\theta} \in \Omega^*(M/G) \mid q^*\bar{\theta} \in \mathcal{I}, \quad q : M \to M/G \}.$$ 

**Reconstruction:** Given a solution $\bar{s} : N \to M/G$ of $\mathcal{I}/G$, a (local) solution to $\mathcal{I}$ on $M$ can be found by solving a system of equations of Lie type (using one of the 3 methods outlined in the example). If the group $G$ is solvable, these can be integrated by quadratures.

In our example

$$I = \text{span}\{dv - v_x dx, dv_x - v_{xx} dx\} ; \quad \mathcal{I} = \langle \theta, \theta_x, d\hat{\xi} \wedge dx \rangle,$$
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$

**Congruence:** Two integral manifolds $s_1, s_2 : N \to M$ with $N$ connected, are congruent if and only if the projection $q \circ s_1 = q \circ s_2$ are the same solutions to the quotient

$$\bar{\mathcal{I}} = \{ \bar{\theta} \in \Omega^*(M/G) \mid q^*\bar{\theta} \in \mathcal{I}, \ q : M \to M/G \}.$$

**Reconstruction:** Given a solution $\bar{s} : N \to M/G$ of $\mathcal{I}/G$, a (local) solution to $\mathcal{I}$ on $M$ can be found by solving a system of equations of Lie type (using one of the 3 methods outlined in the example). If the group $G$ is solvable, these can be integrated by quadratures.

In our example

$$I = \text{span}\{dv - v_x dx, dv_x - v_{xx} dx\} ; \quad \mathcal{I} = \langle \theta, \theta_x, d\hat{\xi} \wedge dx \rangle$$

and

$$\mathcal{I}/G = \text{span}\{d\hat{\xi} \wedge dx\}$$
**Theorem:** Let $G$ act freely and regularly on $M$, and as a symmetry of Pfaffian system $\mathcal{I}$ generated by $I \subset T^*M$ satisfying

$$\Gamma_G \cap \text{annihilator}(I) = 0.$$ 

**Congruence:** Two integral manifolds $s_1, s_2 : N \to M$ with $N$ connected, are congruent if and only if the projection $q \circ s_1 = q \circ s_2$ are the same solutions to the quotient

$$\bar{\mathcal{I}} = \{ \bar{\theta} \in \Omega^*(M/G) \mid q^* \bar{\theta} \in \mathcal{I}, \quad q : M \to M/G \}.$$ 

**Reconstruction:** Given a solution $\bar{s} : N \to M/G$ of $I/G$, a (local) solution to $\mathcal{I}$ on $M$ can be found by solving a system of equations of Lie type (using one of the 3 methods outlined in the example). If the group $G$ is solvable, these can be integrated by quadratures.

In our example

$$I = \text{span}\{dv - v_x dx, dv_x - v_{xx} dx\} ; \quad \mathcal{I} = \langle \theta, \theta_x, d\hat{\xi} \wedge dx \rangle$$

and

$$\mathcal{I}/G = \text{span}\{d\hat{\xi} \wedge dx\}$$

A solution to the quotient is just prescribing $\hat{\xi} = G(x)$. 
Part 2

The Cauchy Problem
Let’s continue the example with

\[ G = \{ (a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R} \} \]
Let’s continue the example with

\[ G = \{ (a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R} \} \]

now acting diagonally on \( J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \),

\[
(a, b) \cdot (x, v, v_x, v_{xx}; y, w, w_y, w_{yy}) = (x, av + b, av_x, av_{xx}; y, aw - b, aw_y aw_{yy})
\]
Let’s continue the example with

\[ G = \{ (a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R} \} \]

now acting diagonally on \( J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \),

\[ (a, b) \cdot (x, \nu, \nu_x, \nu_{xx}; y, w, w_y, w_{yy}) = (x, av + b, av_x, av_{xx}; y, aw - b, aw_yaw_{yy}) \]

The invariants are (using the action commutes with total differentiation),

\[ x, y, u = x - \frac{\nu + w}{\nu_x}, \quad u_x = D_x(u) = \frac{(\nu + w)\nu_{xx}}{\nu_x^2}, \quad u_y = D_y(u) = -\frac{w_y}{\nu_x}, \ldots \]
Let's continue the example with

\[ G = \{ (a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R} \} \]

now acting diagonally on \( J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \),

\[(a, b) \cdot (x, \nu, \nu_x, \nu_{xx}; y, w, w_y, w_{yy}) = (x, av + b, av_x, av_{xx}; y, aw - b, aw_y aw_{yy})\]

The invariants are (using the action commutes with total differentiation),

\[ x, y, u = x - \frac{v + w}{v_x}, u_x = D_x(u) = \frac{(v + w)v_{xx}}{v_x^2}, u_y = D_y(u) = -\frac{w_y}{v_x}, \ldots \]

We also have the relation (syzygy),

\[ u_{xy} = D_y(u_x) = D_x(-\frac{w_y}{v_x}) = \frac{w_y v_{xx}}{v_x^2} = \frac{u_x u_y}{u - x} \]
Let’s continue the example with

\[ G = \{ (a, b) \mid a \in \mathbb{R}^*, b \in \mathbb{R} \} \]

now acting diagonally on \( J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \),

\[ (a, b) \cdot (x, \nu, \nu_x, \nu_{xx}; y, w, w_y, w_{yy}) = (x, av + b, av_x, av_{xx}; y, aw - b, aw_y aw_{yy}) \]

The invariants are (using the action commutes with total differentiation),

\[ x, y, u = x - \frac{\nu + w}{\nu_x}, \quad u_x = D_x(u) = \frac{(\nu + w)\nu_{xx}}{\nu_x^2}, \quad u_y = D_y(u) = -\frac{w_y}{\nu_x}, \ldots \]

We also have the relation (syzygy),

\[ u_{xy} = D_y(u_x) = D_x(-\frac{w_y}{\nu_x}) = \frac{w_y \nu_{xx}}{\nu_x^2} = \frac{u_x u_y}{u - x} \]

So \( J^2 \times J^2/G_{\text{diag}} \) gives rise to the PDE in the equation above.
We have the projection maps coming from each $J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})/G$,

$$
q_1(x, v, v_x, v_{xx}) = \left( x, \hat{\xi} = \frac{v_{xx}}{v_x} \right), \quad q_2(y, w, w_y, w_{yy}) = \left( y, \hat{\xi} = \frac{w_{yy}}{w_y} \right)
$$
We have the projection maps coming from each $J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})/G$,

$$q_1(x, v, v_x, v_{xx}) = \left( x, \hat{\xi} = \frac{v_{xx}}{v_x} \right), \quad q_2(y, w, w_y, w_{yy}) = \left( y, \bar{\xi} = \frac{w_{yy}}{w_y} \right)$$

Producing the big diagram

\[
\begin{array}{c}
\left( x, v, v_x, v_{xx}, v_{xxx} ; \ y, w, w_y, w_{yy} \right) \\
\downarrow q \circ G_{\text{diag}} \\
\left( \ x, y, u = x - \frac{v+w}{v_x}, u_x = \frac{(v+w)v_{xx}}{v_x^2}, u_y = -\frac{w_y}{v_x}, u_{xx} = D_x(u_x), u_{yy} = -\frac{w_{yy}}{v_x} \right)
\end{array}
\]

where $(q_1, q_2) = (p_1 \circ q \circ G_{\text{diag}}, p_2 \circ q \circ G_{\text{diag}})$.

Remark: The maps $p_i$ are not group quotients.
We have the projection maps coming from each $J^2(R, R) \rightarrow J^2(R, R)/G$,

$$
q_1(x, v, v_x, v_{xx}) = \left( x, \hat{\xi} = \frac{v_{xx}}{v_x} \right), \quad q_2(y, w, w_y, w_{yy}) = \left( y, \hat{\xi} = \frac{w_{yy}}{w_y} \right)
$$

Producing the big diagram

where $(q_1, q_2) = (p_1 \circ q_{G_{\text{diag}}}, p_2 \circ q_{G_{\text{diag}}})$.

Remark: The maps $p_i$ are not group quotients.
Take the Cauchy data $S$ for $u_{xy} = (u_x u_y)(u - x)^{-1}$ along $y = x$

$u = f(x), u_x = g(x), u_y = f'(x) - g(x), u_{xx} = g' - \frac{g(f' - g)}{f - x}, u_{yy} = f'' - g' - \frac{g(f' - g)}{f - x}$
Take the Cauchy data $S$ for $u_{xy} = (u_x u_y)(u - x)^{-1}$ along $y = x$

$u = f(x), u_x = g(x), u_y = f'(x) - g(x), u_{xx} = g' - \frac{g(f' - g)}{f - x}, u_{yy} = f'' - g' - \frac{g(f' - g)}{f - x}$

Using the lower part of the previous diagram

$$\left( x, y, u = x - \frac{v+w}{v_x}, u_x = \frac{(v+w)v_{xx}}{v_x^2}, u_y = -\frac{w_y}{v_x}, u_{xx} = D_x(u_x), u_{yy} = -\frac{w_{yy}}{v_x} \right)$$

$\textbf{P}_1$

$$\left( x, \hat{\xi} = \frac{v_{xx}}{v_x} = \frac{u_x}{x-u}, \right)$$

$\textbf{P}_2$

$$\left( y, \tilde{\xi} = \frac{w_{yy}}{w_y} = \frac{u_{yy}}{u_y} \right)$$
Take the Cauchy data $S$ for $u_{xy} = (u_x u_y) (u - x)^{-1}$ along $y = x$

\[ u = f(x), \quad u_x = g(x), \quad u_y = f'(x) - g(x), \quad u_{xx} = g' - \frac{g(f' - g)}{f - x}, \quad u_{yy} = f'' - g' - \frac{g(f' - g)}{f - x} \]

Using the lower part of the previous diagram

\[
\begin{align*}
(x, y, u = x - \frac{v+w}{v_x}, u_x = \frac{(v+w)v_{xx}}{v_x^2}, u_y = -\frac{w_y}{v_x}, u_{xx} = D_x(u_x), u_{yy} = -\frac{w_{yy}}{v_x})
\end{align*}
\]

Project Cauchy data $S$ using $p_1$ and $p_2$ to the curves

\[
S_1 = p_1(S) = \left(x, \hat{\xi} = \frac{u_x}{x-u} = \frac{g(x)}{x-f(x)}\right)
\]

\[
S_2 = p_2(S) = \left(y, \tilde{\xi} = \frac{u_{yy}}{u_y} = \log(f'(y) - g(y))' - \frac{g(y)}{y-f(y)}\right)
\]
We now apply the reconstruction problem to

\[ S_1 = \left( x, \hat{\xi} = \frac{u_x}{x - u} = \frac{g(x)}{x - f(x)} \right) \]

\[ S_2 = \left( y, \bar{\xi} = \frac{u_{yy}}{u_y} = \log(f'(y) - g(y))' - \frac{g(y)}{y - f(y)} \right) \]
We now apply the reconstruction problem to

\[ S_1 = \left( x, \hat{\xi} = \frac{u_x}{x - u} = \frac{g(x)}{x - f(x)} \right) \]

\[ S_2 = \left( y, \tilde{\xi} = \frac{u_{yy}}{u_y} = \log(f'(y) - g(y))' - \frac{g(y)}{y - f(y)} \right) \]

Using the points

\[(x = 0, v = 0, v_x = 1); (y = 0, w = -f(0), w_y = g(0) - f'(0))\]
We now apply the reconstruction problem to

\[ S_1 = \left( x, \hat{\xi} = \frac{u_x}{x - u} = \frac{g(x)}{x - f(x)} \right) \]

\[ S_2 = \left( y, \hat{\xi} = \frac{u_{yy}}{u_y} = \log(f'(y) - g(y))' - \frac{g(y)}{y - f(y)} \right) \]

Using the points

\((x = 0, v = 0, v_x = 1); (y = 0, w = -f(0), w_y = g(0) - f'(0))\)

the solution to the Lie equation determining \(v\) gives (with \(G(x) = \frac{g(x)}{x - f(x)}\))

\[ v(x) = \int_0^x \left( \exp(\int_0^s \frac{g(t)}{t - f(t)} dt) \right) ds \]
We now apply the reconstruction problem to

\[ S_1 = \left(x, \hat{\xi} = \frac{u_x}{x - u} = \frac{g(x)}{x - f(x)}\right) \]

\[ S_2 = \left(y, \tilde{\xi} = \frac{u_{yy}}{u_y} = \log(f'(y) - g(y))' - \frac{g(y)}{y - f(y)}\right) \]

Using the points

\((x = 0, v = 0, v_x = 1); (y = 0, w = -f(0), w_y = g(0) - f'(0))\)

the solution to the Lie equation determining \(v\) gives (with \(G(x) = \frac{g(x)}{x - f(x)}\))

\[ v(x) = \int_0^x \left( \exp\left( \int_0^s \frac{g(t)}{t - f(t)} dt \right) \right) ds \]

While solving the Lie equation determining \(w\) gives

\[ w(y) = \int_0^y \left( (g(s) - f'(s)) \exp\left( \int_0^s \frac{g(t)}{t - f(t)} dt \right) \right) ds - f(0) \]
With $G(t) = g(t)(t - f(t))^{-1}$
With \( G(t) = g(t)(t - f(t))^{-1} \)

\[
\begin{align*}
\nu(x) &= \int_0^x \left( e^{\int_0^s G(t)dt} \right) ds \\
\omega(y) &= \int_0^y \left( (g(s) - f'(s)) e^{\int_0^s G(t)dt} \right) ds - f(0) \\
&= \int_0^y \left( (s - f(s)) G(s) - f'(s) e^{\int_0^s G(t)dt} \right) ds - f(0) \\
&= -\int_0^y \left( e^{\int_0^s G(t)dt} \right) ds + (y - f(y)) \int_0^y e^{\int_0^s G(t)dt} ds
\end{align*}
\]
With \( G(t) = g(t)(t - f(t))^{-1} \)

\[
\begin{align*}
v(x) &= \int_0^x \left( e^{\int_0^s G(t) \, dt} \right) \, ds \\
w(y) &= \int_0^y \left( (g(s) - f'(s)) e^{\int_0^s G(t) \, dt} \right) \, ds - f(0) \\
&= \int_0^y \left( (s - f(s)) G(s) - f'(s) e^{\int_0^s G(t) \, dt} \right) \, ds - f(0) \\
&= -\int_0^y \left( e^{\int_0^s G(t) \, dt} \right) \, ds + (y - f(y)) \int_0^y e^{\int_0^s G(t) \, dt} \, ds 
\end{align*}
\]

Combining these together via the diagonal quotient map \( q_{G_{\text{diag}}} \) gives

\[
\begin{align*}
u(x) &= \frac{v(x) + w(y)}{v_x} \\
&= x + (f(y) - y) e^{\int_x^y G(t) \, dt} + e^{-\int_x^x G(t) \, dt} \left( \int_x^y e^{\int_0^s G(t) \, dt} \, ds \right) .
\end{align*}
\]
With \( G(t) = g(t)(t - f(t))^{-1} \)

\[
\nu(x) = \int_0^x \left( e^{\int_0^s G(t) dt} \right) ds
\]

\[
\omega(y) = \int_0^y \left( (g(s) - f'(s)) e^{\int_0^s G(t) dt} \right) ds - f(0)
\]

\[
= \int_0^y \left( (s - f(s)) G(s) - f'(s) e^{\int_0^s G(t) dt} \right) ds - f(0)
\]

\[
= - \int_0^y \left( e^{\int_0^s G(t) dt} \right) ds + (y - f(y)) \int_0^y e^{\int_0^s G(t) dt} ds
\]

Combining these together via the diagonal quotient map \( q_{G_{\text{diag}}} \) gives

\[
u(x) + \omega(y)
\]

\[
u_x
\]

\[
= x + (f(y) - y) e^{\int_x^y G(t) dt} + e^{- \int_0^x G(t) dt} \left( \int_x^y e^{\int_0^s G(t) dt} ds \right).
\]

Which solves the Cauchy problem \( u = f(x), \ u_x = g(x) \) along \( y = x \).
With $G(t) = g(t)(t - f(t))^{-1}$

$$v(x) = \int_0^x \left( e^{\int_0^s G(t) dt} \right) ds$$

$$w(y) = \int_0^y \left( (g(s) - f'(s)) e^{\int_0^s G(t) dt} \right) ds - f(0)$$

$$= \int_0^y \left( (s - f(s)) G(s) - f'(s) \right) e^{\int_0^s G(t) dt} ds - f(0)$$

$$= - \int_0^y \left( e^{\int_0^s G(t) dt} \right) ds + (y - f(y)) \int_0^y e^{\int_0^s G(t) dt} ds$$

Combining these together via the diagonal quotient map $q_{G_{\text{diag}}}$ gives

$$u = x - \frac{v(x) + w(y)}{v_x}$$

$$= x + (f(y) - y) e^{\int_x^y G(t) dt} + e^{-\int_0^x G(t) dt} \left( \int_x^y e^{\int_0^s G(t) dt} ds \right).$$

Which solves the Cauchy problem $u = f(x), u_x = g(x)$ along $y = x$. **Remark:**

Analogue to D’Alembert’s formula for the Wave equation.
Darboux Integrability

The equation \( u_{xy} = \frac{u_x u_y}{u-x} \) is Darboux Integrable (Admits intermediate integrals).
Darboux Integrability

The equation \( u_{xy} = \frac{u_x u_y}{u-x} \) is Darboux Integrable (Admits intermediate integrals).

\[
\frac{d}{dy} \hat{\xi} = \frac{d}{dy} \frac{u_x}{u-x} = \frac{u_{xy}}{u-x} - \frac{u_x u_y}{(u-x)^2} = 0
\]
The equation \( u_{xy} = \frac{u_x u_y}{u - x} \) is Darboux Integrable (Admits intermediate integrals).

\[
\frac{d}{dy} \hat{\xi} = \frac{d}{dy} \frac{u_x}{u - x} = \frac{u_{xy}}{u - x} - \frac{u_x u_y}{(u - x)^2} = 0
\]

\[
\frac{d}{dx} \hat{\xi} = \frac{d}{dx} \frac{u_{yy}}{u_y} = \frac{u_{xxy}}{u_y} - \frac{u_{yy} u_{xy}}{u_y^2} = \frac{u_x u_{yy}}{u - x} - \frac{u_{yy} u_x u_y}{u - x} = 0
\]

We call \( G \) the Vessiot group. Theorem: Let \( I \) be a Darboux integrable system. If the Vessiot group \( G \) for \( I \) is solvable then the Cauchy problem for \( I \) can be solved by quadratures.
**Darboux Integrability**

The equation \( u_{xy} = \frac{u_x u_y}{u-x} \) is **Darboux Integrable** (Admits intermediate integrals).

\[
\frac{d}{dy} \hat{\xi} = \frac{d}{dy} \frac{u_x}{u-x} = \frac{u_{xy}}{u-x} - \frac{u_x u_y}{(u-x)^2} = 0
\]

\[
\frac{d}{dx} \hat{\xi} = \frac{d}{dx} \frac{u_{yy}}{u_y} = \frac{u_{xxy}}{u_y} - \frac{u_{yy} u_{xy}}{u^2} = \frac{u_x u_{yy}}{u-x} - \frac{u_{yy} u_x u_y}{u-x} = 0
\]

If \( \mathcal{I} \) is a Darboux integrable system then \( \mathcal{I} \) has a quotient representation.

\[
(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)
\]

\[
\text{qG}_{\text{diag}}
\]

\[
(\mathcal{I}, M)
\]
Darboux Integrability

The equation \( u_{xy} = \frac{u_x u_y}{u-x} \) is Darboux Integrable (Admits intermediate integrals).

\[
\frac{d}{dy} \xi = \frac{d}{dy} \frac{u_x}{u-x} = \frac{u_{xy}}{u-x} - \frac{u_x u_y}{(u-x)^2} = 0
\]

\[
\frac{d}{dx} \xi = \frac{d}{dx} \frac{u_{yy}}{u_y} = \frac{u_{xxy}}{u_y} - \frac{u_{yy} u_{xy}}{u_y^2} = \frac{u_x u_{yy}}{u-x} - \frac{u_{yy} u_x u_y}{u-x} = 0
\]

If \( \mathcal{I} \) is a Darboux integrable system then \( \mathcal{I} \) has a quotient representation.

\[
(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)
\]

\[
\downarrow \quad \text{q}_{G_{\text{diag}}}
\]

\[
(\mathcal{I}, M)
\]

\( G \) is a symmetry group of \( \mathcal{I}_i \) acting on \( M_i \), \( \text{q}_{G_{\text{diag}}} \) is the quotient by the diagonal action. See Anderson, Fels, Vasilliou Adv. Math. 2007
Darboux Integrability

The equation $u_{xy} = \frac{u_x u_y}{u-x}$ is Darboux Integrable (Admits intermediate integrals).

\[
\frac{d}{dy} \hat{\xi} = \frac{d}{dy} \frac{u_x}{u-x} = \frac{u_{xy}}{u-x} - \frac{u_x u_y}{(u-x)^2} = 0
\]

\[
\frac{d}{dx} \hat{\xi} = \frac{d}{dx} \frac{u_{yy}}{u_y} = \frac{u_{xyy}}{u_y} - \frac{u_{yy} u_{xy}}{u^2} = \frac{u_x u_{yy}}{u-x} - \frac{u_{yy} u_x u_y}{u-x} = 0
\]

If $\mathcal{I}$ is a Darboux integrable system then $\mathcal{I}$ has a quotient representation.

\[
(I_1 + I_2, M_1 \times M_2) \xrightarrow{\text{q}_{G_{\text{diag}}}} (\mathcal{I}, M)
\]

$G$ is a symmetry group of $\mathcal{I}_i$ acting on $M_i$, $\text{q}_{G_{\text{diag}}}$ is the quotient by the diagonal action. See Anderson, Fels, Vasiliou Adv. Math. 2007

We call $G$ the Vessiot group.
Darboux Integrability

The equation \( u_{xy} = \frac{u_x u_y}{u - x} \) is Darboux Integrable (Admits intermediate integrals).

\[
\frac{d}{dy} \hat{\xi} = \frac{d}{dy} \frac{u_x}{u - x} = \frac{u_{xy}}{u - x} - \frac{u_x u_y}{(u - x)^2} = 0
\]

\[
\frac{d}{dx} \hat{\xi} = \frac{d}{dx} \frac{u_{yy}}{u_y} = \frac{u_{xxy}}{u_y} - \frac{u_{yy} u_{xy}}{u_y^2} = \frac{u_x u_{yy}}{u - x} - \frac{u_{yy} u_x u_y}{u - x} = 0
\]

If \( \mathcal{I} \) is a Darboux integrable system then \( \mathcal{I} \) has a quotient representation.

\[
(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)
\]

\[
\xrightarrow{\mathbf{q}_{G_{\text{diag}}}}
\]

\[
(\mathcal{I}, M)
\]

\( G \) is a symmetry group of \( \mathcal{I}_i \) acting on \( M_i \), \( \mathbf{q}_{G_{\text{diag}}} \) is the quotient by the diagonal action. See Anderson, Fels, Vasiliou Adv. Math. 2007

We call \( G \) the Vessiot group.

**Theorem:** Let \( \mathcal{I} \) be a Darboux integrable system. If the Vessiot group \( G \) for \( \mathcal{I} \) is solvable then the Cauchy problem for \( \mathcal{I} \) can be solved by quadratures.
Proof The proof follows the example.
Proof The proof follows the example.

1 Construct the diagram,

\[(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)\]

\[
\begin{array}{c}
\pi_1^G \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(\mathcal{I}, M) \\
\downarrow \quad \downarrow \quad \downarrow \\
(I_1/G, M_1/G) \quad (I_2/G, M_2/G) \\
\end{array}
\]

\[
\begin{array}{c}
\pi_2^G \\
\end{array}
\]

\[q_{G_{\text{diag}}} \]

Project Cauchy data \(S \subset M\) using \(p_1\) and \(p_2\) which are solutions to \(I_1 \big/ G\), \(S_1 = p_1(S)\), \(S_2 = p_2(S)\).

Solve the reconstruction problem for \(I_1 \big/ q_{G_{\text{diag}}} - 1\) \((S_1)\) and \(I_2 \big/ q_{G_{\text{diag}}} - 1\) \((S_2)\) using quadratures (since \(G\) is solvable).

The solution is the projection of the product of the solutions \(u = q_{G_{\text{diag}}} \big/ \mathcal{N} \times \mathcal{N} \big/ \mathcal{N}\).
Proof The proof follows the example.

1. Construct the diagram,

\[(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)\]

\[
\begin{array}{c}
\pi_1^G \\
\downarrow \\
(\mathcal{I}, M) \\
\downarrow \\
(\mathcal{I}_1/G, M_1/G) \quad \quad (\mathcal{I}_2/G, M_2/G) \\
\pi_2^G \\
\end{array}
\]

\[
\begin{array}{c}
p_1 \\
\downarrow \\
\pi G_{\text{diag}} \\
\downarrow \\
p_2 \\
\end{array}
\]

2. Project Cauchy data \( S \subset M \) using \( p_1 \) and \( p_2 \) which are solutions to \( \mathcal{I}_a/G \),

\[
S_1 = p(S), \quad S_2 = p(S)
\]
**Proof** The proof follows the example.

1. Construct the diagram,

   $$(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)$$

   $$\xymatrix{(\mathcal{I}_1/G, M_1/G) \ar[r]_{\pi_1^G} \ar[d]_{\pi_{G,\text{diag}}} \ar[l]_{p_1} & (\mathcal{I}, M) \ar[d]^{\pi_2^G} \ar[l]_{p_2} & (\mathcal{I}_2/G, M_2/G) \ar[l]_{p_2}}$$

2. Project Cauchy data $S \subset M$ using $p_1$ and $p_2$ which are solutions to $\mathcal{I}_a/G$,

   $$S_1 = p(S), \quad S_2 = p(S)$$

3. Solve the reconstruction problem for $\mathcal{I}_1|_{q_1^{-1}(S_1)}$ and $\mathcal{I}_2|_{q_2^{-1}(S_2)}$ using quadratures (since $G$ is solvable).
**Proof** The proof follows the example.

1. Construct the diagram,

\[(\mathcal{I}_1 + \mathcal{I}_2, M_1 \times M_2)\]

\[
\begin{array}{c}
\pi_1^G \\
\downarrow \\
(\mathcal{I}_1/M_1/G) \\
\uparrow \\
\pi_2^G \\
\downarrow \\
(\mathcal{I}_2/M_2/G)
\end{array}
\]

\[
\begin{array}{c}
\mathbf{p}_1 \\
\mathbf{q}_{G_{\text{diag}}} \\
\mathbf{p}_2
\end{array}
\]

2. Project Cauchy data \( S \subset M \) using \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) which are solutions to \( \mathcal{I}_a/G \),

\[
S_1 = \mathbf{p}(S), \quad S_2 = \mathbf{p}(S)
\]

3. Solve the reconstruction problem for \( \mathcal{I}_1|_{q_1^{-1}(S_1)} \) and \( \mathcal{I}_2|_{q_2^{-1}(S_2)} \) using quadratures (since \( G \) is solvable).

4. The solution is the projection of the product of the solutions \( N_a \subset q_{a^{-1}}(S_a) \),

\[
u = q_{G_{\text{diag}}} (N_1 \times N_2).
\]
The End.