Real Analysis Prelim Written Exam Spring 2017 [with revision]

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Given $\varepsilon > 0$, construct an open set $U \subset \mathbb{R}$ containing $\mathbb{Q}$ and with Lebesgue measure less than $\varepsilon$.

[2] Give an example of a sequence $\{f_n\}$ of continuous functions on $[0, 1]$ such that $\lim_n f_n(x) = 0$ for all $x \in [0, 1]$, but $\int_0^1 f_n(t) \, dt = 1$ for all $n$.

[3] Suppose that $f \in L^1(\mathbb{R})$ and $\int_a^b f(x) \, dx = 0$ for all $a, b \in \mathbb{R}$. Show that $f(x) = 0$ almost everywhere.

[4] [revised] Let $f \in L^2[0, 1]$ with distributional derivative $f' \in L^2[0, 1]$. Show that there is a constant $C$ such that $|f(x) - f(y)| < C \cdot |x - y|^{1/2}$ for $x, y \in [0, 1]$.

[5] Let $C = \{v = (v_1, v_2, \ldots) \in \ell^2 : |v_n| \leq \frac{1}{n}\} \subset \ell^2$. Show that $C$ is compact.

[6] Let $E$ be a Lebesgue measurable subset of $[0, 1]$, and let $f(x) = \int_E \sin(tx) \, dt$. Show that $f(x)$ is continuous.

[7] For $1 < p < \infty$, $f \in L^p(\mathbb{R})$, and $g \in L^1(\mathbb{R})$, show that $|f \ast g|_{L^p} \leq |f|_{L^p} \cdot |g|_{L^1}$.

[8] Show that $C^1[a, b]$ with norm $|f|_{C^1} = \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|$ is a Banach space.