Real Analysis Preliminary Exam
April 22, 2010

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

(12) 1. Let \( f \in L^1(\mathbb{R}) \). Show that \( \int_{-x}^{x} f(t)dt \to \int_{-\infty}^{\infty} f(t)dt \) as \( x \to \infty \).

(12) 2. Let \( f: [0,1] \to [0,1] \) be a nondecreasing continuous function satisfying \( f(0) = 0 \) and \( f(1) = 1 \). Prove or give a counterexample: \( \int_{0}^{1} f'(x)dx = 1 \).

(16) 3. Prove or disprove:
   (8) a. Every locally compact metric space is complete.
   (8) b. Every complete metric space is locally compact.

(12) 4. Let \( f: \mathbb{R} \to \mathbb{R} \) be absolutely continuous, and assume that \( f' \in L^1(\mathbb{R}) \) and that \( f(0) = 0 \). Show that the following limit exists, and compute its value.
   \[
   \lim_{x \to 0^+} x^{-2/3} f(x)
   \]

(16) 5. Suppose that \( f: \mathbb{R} \to \mathbb{C} \) is locally integrable (i.e., \( f \cdot \chi_k \in L^1(\mathbb{R}) \), for every compact \( K \)), and suppose that \( g: \mathbb{R} \to \mathbb{C} \) is continuous. Prove or disprove.
   (8) a. If \( g \) has compact support, then \( f \ast g \) is uniformly continuous.
   (8) b. If \( g \) is bounded and if \( f \) has compact support, then \( f \ast g \) is uniformly continuous.
6. Consider a function $f: \mathbb{R} \to \mathbb{R}$ which is periodic with period one and which satisfies

$$f(x) = \chi_{(0,1/2)}(x) - \chi_{(-1/2,0)}(x) \quad \text{for} \quad |x| < 1/2.$$ 

a. Compute the Fourier series for $f$.

b. Use your result from item (a) to compute $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

7. Assume that $E$ is a bounded Lebesgue measurable subset of $\mathbb{R}$, and let

$$f(t) = \int_E \cos(tx) \, dx, \quad t \in \mathbb{R}.$$ 

Prove or disprove:

a. $f$ has compact support.

b. $f$ is differentiable.