Real Analysis Prelim Written Exam Fall 2018

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Let $f$ be a continuous function on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. Show that 
\[ \{x \in [0, 1] : \frac{1}{3} < f(x) < \frac{2}{3} \} \]
has positive Lebesgue measure.

[2] Show that the closed unit ball in $\ell^2$ is not compact (although it is closed and bounded).

[3] Compute  
\[ \int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^2 \, dx \]

[4] For a Lebesgue-measurable subset $E$ of $\mathbb{R}$ with finite Lebesgue measure, show that
\[ \lim_{t \to +\infty} \int_E \sin(tx) \, dt = 0. \]

[5] Let $f \in C_c^0(\mathbb{R})$. Show that for every $g \in L^1(\mathbb{R})$, 
\[ \lim_{t \to +\infty} \int_{\mathbb{R}} f(x + t) g(x) \, dx = 0. \]

[6] Show that $C^1[a, b]$ with norm $\|f\|_{C^1} = \sup_{x \in [a, b]} |f(x)| + \sup_{x \in [a, b]} |f'(x)|$ is a Banach space.

[7] Let $S, T$ be self-adjoint compact operators on a Hilbert space $V$, with $ST = TS$. Show that there is an orthonormal basis for $V$ consisting of simultaneous eigenvectors for $S$ and $T$. 