Real Analysis Preliminary Exam
August 29, 2016

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Please use separate bluebooks for parts A and B, and clearly label each bluebook, indicating its contents.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

Part A

(12) 1. Assume that $S$ is a Lebesgue measurable subset of $\mathbb{R}$ and that $S$ has positive measure. Show the existence of two distinct points $x \in S$ and $y \in S$ such that $x - y$ is rational.

(15) 2. 
   (10) a. Construct a continuous, non-decreasing, surjective function $f:[0,1] \to [0,1]$ such that $f^{-1}(y)$ contains an open interval for every dyadic rational number $y \in (0,1)$. (A dyadic rational number is a rational number whose denominator is a power of two.)
   (5) b. Does your function satisfy $\int_0^1 f'(x)dx = f'(1) - f'(0) = 1$? Why or why not? Explain why the fundamental theorem of calculus may or may not be applied here.

(12) 3. Let $X$ be a complete real inner product space such that every closed and bounded set is compact. Prove or disprove: $X$ is finite dimensional.

(12) 4. Suppose that $F:[a,b] \to \mathbb{R}$ and $G:[a,b] \to \mathbb{R}$ are absolutely continuous. Show that $FG$ is absolutely continuous.
Part B

(17) 5.

(5) a. State the Riemann-Lebesgue Lemma for functions in $L^1([0,1])$.

(6) b. Using your statement in part (a), prove the following statement. If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable periodic function with period 1, then

$$\lim_{n \to \infty} \int_0^1 f(x) \sin 2\pi nx \, dx = 0,$$

where $n$ is an integer.

(6) c. Is the statement in part (b) true if $f$ is not periodic? Why or why not?

(20) 6.

(5) a. State Fubini’s Theorem for $L^1$ functions on measure spaces. Be sure to state the hypotheses and conclusions fully and precisely.

(5) b. Show that

$$\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx = \frac{\pi}{4}.$$

Hint: \( \frac{d}{dy} \left( \frac{y}{x^2 + y^2} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \)

(5) c. Show that

$$\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \right) \, dy = -\int_0^1 \left( \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \right) \, dx.$$

(5) d. Explain why parts (b) and (c) do not provide a counterexample to Fubini’s Theorem.

(12) 7. Using an appropriate Fourier series, show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).