Real Analysis Preliminary Exam
August 29, 2013

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Please use separate bluebooks for parts A and B, and clearly label each bluebook, indicating its contents.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

The weight of each problem is indicated to the left of the problem.

Part A

(14) 1. Let $E$ and $F$ be Lebesgue measurable subsets of $\mathbb{R}$, with $E \subseteq F$. Suppose that $F$ is Borel measurable, but suppose that $E$ is not Borel measurable. Prove or disprove: $F \cap E^c$ is uncountable.

(14) 2. 
   a. Construct an example of a set $U \subset [0,1]$ such that $U$ is open and dense and such that $m(U) < 1$, where $m$ denotes Lebesgue measure.
   b. Let $E$ be a countable dense subset of $[0,1]$, and let $U$ be an open dense subset of $[0,1]$ satisfying $m(U) < 1$. Prove or disprove: $E \cap U^c \neq \emptyset$.

(15) 3. 
   a. State Parseval’s identity for Fourier series of functions on $[0,1]$.
   b. Using Parseval’s identity, show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(15) 4. 
   a. State the Riemann-Lebesgue Lemma for Fourier transforms of functions on $\mathbb{R}$.
   b. Let $E \subset \mathbb{R}$ have finite Lebesgue measure. Show that $f(t) = \int_E \sin tx \, dx$ is a continuous function.
   c. How differentiable is $f$ if $E$ is also bounded?
Part B

(14) 5. Suppose that $F: [a, b] \rightarrow \mathbb{R}$ and $G: [a, b] \rightarrow \mathbb{R}$ are absolutely continuous. Show that $FG$ is absolutely continuous.

(14) 6. Let $f_n$ be a sequence of nonnegative functions in $L^1((0,1))$ such that $\|f_n\|_1 = 1$, for all $n$, and $f_n \rightarrow 0$ a.e. as $n \rightarrow \infty$. Prove that $\int f_n \rightarrow 0$ as $n \rightarrow \infty$.

(14) 7. Let $f, g \in L^1(\mathbb{R})$, and let $h = f \ast g$. Suppose that $\int yf(y)dy = 0$ and that $\int yg(y)dy = 0$. Prove that $\int yh(y)dy = 0$. 