Instructions

- Solve AT MOST 5 of the following problems. It is not an option to try for partial credit on all eight problems: solutions to at most 5 problems will be collected at the end of the exam.
- Solutions to different problems must be written on different sheets of paper because different people may grade different problems. You will find sheets of paper for this use on your desk. Bluebooks may be used for scratch notes and will not be collected.
- Remember to mark each page you turn in with your “codename”. DO NOT use your actual name.
- Each problem is worth 20 points, so 100 points is the maximum possible to score on this examination.
- No books, papers, or electronic devices may be used in this examination. In particular, you may not use your own scratch paper.
- All answers must be complete and correct. Partial credit will be awarded sparingly. The scorers must not be expected to supply any missing parts of any answer. Solutions of problems not laid out clearly will be penalized heavily. Do not make additional hypotheses that trivialize a problem.
Problem 1
Let \( f_n : I \rightarrow \mathbb{R} \) be a sequence of real functions defined on \( I = (0, 1) \). For the following statements, determine whether they are true or false. If true, provide a proof. If false, give a counterexample.

(i) (5 points) If \( f_n \) is a monotonically decreasing sequence of nonnegative functions, and \( \|f_n\|_{L^1(I)} \rightarrow 0 \), then \( f_n(x) \) converges to 0 for every \( x \in I \).

(ii) (5 points) If \( f_n(x) \) converges to 0 for every \( x \), then for some set \( A \subset I \) of (Lebesgue) measure greater than \( 1/2 \), holds: \( \|f_n\|_{L^1(A)} \rightarrow 0 \).

(iii) (5 points) If \( \|f_n\|_{L^1(I)} \rightarrow 0 \), then for some \( x \in I \), \( f_n \) converges.

(iv) (5 points) If \( f_n \) is a monotonically decreasing sequence of nonnegative functions, and \( \|f_n\|_{L^1(I)} \rightarrow 0 \), then \( f_n \) converges to 0 almost everywhere in \( I \).

Problem 2
Let \( f \in L^2(\mathbb{R}^n) \). Prove that:
\[
\lim_{r \to \infty} r^{-n/2}\|f\|_{L^1(B_r)} = 0,
\]
where \( B_r \) denotes the ball \( \{x \in \mathbb{R}^n; \|x\| < r\} \).

Problem 3
Let \( (\mathbb{X}, \mathcal{M}, \mu) \) be a space with a signed measure \( \mu \) (that is, the set function \( \mu : \mathcal{M} \rightarrow \mathbb{R} \) on the \( \sigma \)-algebra \( \mathcal{M} \) of subsets of the set \( X \), satisfies \( \mu(\emptyset) = 0 \) and is countably additive). Prove that \( \mu \) must be bounded, that is:
\[
\exists n > 0 \quad \forall A \in \mathcal{M} \quad |\mu(A)| \leq n.
\]

Problem 4
Let \( H \) be a Hilbert space.

(i) (10 points) Show that no orthonormal basis of \( H \) can be its Hamel basis (that is, a basis of \( H \) as a vector space), unless \( H \) is of finite dimension.

(ii) (10 points) Show that a linear mapping \( P : H \rightarrow H \) is the projection onto some closed subspace of \( H \) if and only if \( P^2 = P \) and \( P \) is self-adjoint.

Problem 5
Let \( \Omega \subset \mathbb{R}^n \) be open, bounded and of (Lebesgue) measure 1. Let \( f \in L^1(\Omega) \). Prove that:
\[
\lim_{p \to 0} \left( \int_{\Omega} |f|^p \right)^{1/p} = \exp \left( \int_{\Omega} \ln |f| \right).
\]

Hint: To prove the inequality \( \leq \), you may first prove that \( \lim_{p \to 0} (|x|^p - 1)/p = \ln |x| \).
Problem 7

(i) (10 points) Let
\[ A(t) = \int_0^1 e^{itu} \frac{1}{\sqrt{1-u}} \, du \]
be a function of \( t \in \mathbb{R} \). Show that \( A(t) = O(t^{-1/2}) \) for \( t \to \infty \).

(ii) (10 points) Let \( \phi \) be the characteristic function of the unit disc in \( \mathbb{R}^2 \) and \( \hat{\phi} \)
be its Fourier transform. Show that \( \hat{\phi}(y) = O(|y|^{-3/2}) \) for \( |y| \to \infty \).

Problem 8

Let \( E, F \) be two Banach spaces and \( T \) a linear operator from \( E \) to \( F \). Prove that in either of the following two cases:

(i) (10 points) For every convergent sequence \( \{x_n\}_{n=1}^{\infty} \), the sequence \( \{Tx_n\}_{n=1}^{\infty} \)
is weakly converging.

(ii) (10 points) \( T \) is continuous from \( E \) with weak topology to \( F \) with weak topology.

\( T \) must be continuous.