Real Analysis Preliminary Examination – Fall 2004

Instructions

No books, papers, or electronic devices may be used in this Examination!

Write only your “codename” on your Blue Books, NOT your actual name!

This Examination has two Parts, A and B. Solutions of problems in separate Parts must be written is separate Blue Books! The two Parts will be scored by different Scorers.

The point value for each problem is shown in square brackets. There are four problems in Part A, worth 200 points, and six problems in Part B, worth 200 points, for a total of 400 points. The passing score will be based on your total score.

You may use the result of one problem in another one, even if you have not solved the problem in which the result you are using is proved.

All answers must be complete and correct. Partial credit will be awarded sparingly. The Scorers must not be expected to supply any missing parts of any answer.

Problems in each Part may be done in any order. You must clearly identify where the parts of your answers are. The Scorers will not search at length for answers that are incomplete. Rather, they will assume that you stopped your attempt, unless you clearly indicate where the answer continues.

If you use a Theorem, state it fully and concisely, or identify it clearly. In either case, verifying hypotheses explicitly is essential. Do not use a Theorem if the result you are proving is used to prove that Theorem!

In all cases, give the essential explanations and justifications. A significant part of your work in each problem is really the determination of the crucial points!

Do not make assumptions that trivialize a problem.
Part A

A01 [50 pts]: Let \((f_n)_{n=0}^\infty\) and \(f\) belong to \(L^4(\mathbb{R})\). \(f_n\)'s are said to converge to \(f\) weakly if for any \(\phi \in L^{4/3}(\mathbb{R})\),

\[
\int_{\mathbb{R}} f_n(x)\phi(x) \, dx \to \int_{\mathbb{R}} f(x)\phi(x) \, dx, \quad n \to \infty.
\]

(i) How is the exponent \(q = 4/3\) correlated to \(p = 4\)?

(ii) Prove the lower semi-continuity property: if \(f_n \to f\) weakly in \(L^4(\mathbb{R})\), then

\[
\|f\|_{L^4} \leq \liminf_{n \to \infty} \|f_n\|_{L^4}.
\]

A02 [50 pts]: A non-empty subset \(\Gamma\) of \(\ell^1\) is said to be uniformly summable if for any \(\epsilon > 0\), there exists an integer \(I\), such that for any \(x = (x_0, x_1, \cdots) \in \Gamma\),

\[
\sum_{i \geq I} |x_i| < \epsilon.
\]

Suppose \(\Gamma = (x^{(n)})_{n=1}^\infty \subset \ell^1\) is bounded (in \(\ell^1\)) and uniformly summable. Show that there exists a subsequence \((x^{(n_k)})_{k=1}^\infty\) and some \(x^* \in \ell^1\), such that the subsequence weakly converges to \(x^*\). That is, for any \(y \in \ell^\infty\),

\[
\langle x^{(n_k)}, y \rangle = \sum_{i=0}^{\infty} x^{(n_k)}_i y_i \to \langle x^*, y \rangle, \quad k \to \infty.
\]

A03 [50 pts]: Suppose that a given sequence of real functions \((f_n)_{n=1}^\infty \subset L^1(0,1)\) satisfies the following uniformity condition: for any \(\epsilon > 0\), there exists some \(M > 0\), such that, for any \(n = 1, 2, \cdots\)

\[
\int_0^1 |f_n(x)| 1_{\{|f_n| > M\}}(x) \, dx < \epsilon,
\]

where \(1_E\) is the indicator function of \(E\), i.e., where \(1_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}\). In addition, suppose \(f_n(x) \to f(x)\) a.e. on \((0,1)\). Show that the pointwise convergence in fact leads to strong convergence:

\[
\|f_n - f\|_{L^1(0,1)} \to 0, \quad n \to \infty.
\]
A04  [50 pts]: Assume that a non-zero function $\psi(x)$ belongs to $C^m(\mathbb{R})$ and is compactly supported. For any two integers $j$ and $k$, define a scaled and shifted function

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).$$

Suppose that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Show that $\psi$ must satisfy the vanishing-moment condition:

$$\int_{\mathbb{R}} \psi(x) x^k dx = 0, \quad k = 0, 1, \ldots, m.$$ 

To make things simpler, let us assume in addition that

$$[\psi(0)][\psi'(0)] \cdots [\psi^{(m)}(0)] \neq 0.$$ 

[Hint: Taylor expansion]
Part B

B01 [50 pts]: First we define the outer measure of a set $E \subseteq \mathbb{R}^n$, denoted $|E|_e$, in terms of the volumes of intervals $I \subseteq \mathbb{R}^n$. An interval is a closed rectangular parallelepiped with edges parallel to the coordinate axes. The volume of an interval $I$, $v(I)$, is the product of its edge lengths. Finally, we set

$$|E|_e := \inf \left\{ \sum_{k=1}^{\infty} v(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k \right\},$$

where the set here consists of all sums resulting from a covering of $E$ by a sequence $\{I_k\}$ of intervals.

Starting with this definition, write an outline of the construction of Lebesgue measure on measurable subsets of $\mathbb{R}^n$ and the development of its properties. Include necessary definitions and statements of Lemmas and Theorems used, but not proofs. Your outline will be scored on aptness, completeness and, e.g., its utility as notes for lectures.

B02 [20 pts]: Define convex function. You may restrict your definition to functions of a single real variable, with due attention to its domain.

(A) Prove that the logarithm of the Gamma function is convex on $(0, \infty)$. The Gamma function may be defined on that interval by $\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt$.

(B) Define absolutely continuous function defined on an interval $I \subseteq \mathbb{R}$ in such a way that unbounded intervals are allowed. This is not the set-function definition! Prove that a positive convex function $f$ that decreases to zero as $x \to +\infty$ is absolutely continuous on an interval $[a, +\infty)$ if $f(a) < +\infty$. A “self-contained” proof is preferred. Just be sure to very carefully state any results that you use without proof.

B03 [40 pts]: The non-centered Hardy–Littlewood maximal function, $f^*(x)$ is given by

$$f^*(x) := \sup \left\{ \frac{1}{|Q|} \int_Q |f(y)| \, dy : Q \text{ is a cube and } x \in Q \right\}$$

(a cube is an interval with equal edges; see B01).

(A) Prove that if $f$ is integrable on $\mathbb{R}^n$ then $|\{x \in \mathbb{R}^n : f^*(x) > \alpha\}| \leq C_n \|f\|_1 / \alpha$. [25 pts]

(B) Redefine $f^*$ for functions defined on $(0, \infty)$. Suppose that $f$ is positive and decreasing and integrable on $(0, 1)$. Calculate $f^*(x)$ for $x > 0$, in terms of $f$. [15 pts]
B04 [40 pts]:
(A) State and prove Hölder’s Inequality. [15 pts]
(B) State the Converse of Hölder’s Inequality and give a valid idea or method for a proof. [10 pts]
(C) For $1 < p \leq \infty$, if $f \in L^p(0, \infty)$ set $Tf(x) := \frac{1}{x} \int_0^x f(t) \, dt$ for $x > 0$. Prove that there exists a real constant $C_p$ such that $\|Tf\|_p \leq C_p \|f\|_p$. Show that this fails when $p = 1$. [15 pts]

B05 [40 pts]:
State Fubini’s Theorem and Tonelli’s Theorem [20 pts].
Use Tonelli’s Theorem, very simple changes of variable (no worse than linear) and antiderivatives to compute $\int_{-\infty}^{\infty} e^{-x^2} \, dx$. Do not use polar coordinates, though starting the “usual” proof is suggested [20 pts].

B06 [10 pts]: Define inner product space, Hilbert space and orthonormal basis.