Please use separate blue books for parts A and B, and clearly label each blue book of its content.

**Most solutions to these problems should be accompanied by proofs.** Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and issues involved. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct.

This exam is closed-book: no notes or outside assistance is permitted.

All problems carry equal weight.
Part A

(1) Consider the mapping \( P : \mathbb{C} \to \mathbb{C} \) given by \( P(z) = z^2 - 2z \).
   a) Show that \( P \) restricted to \( \mathbb{C} \setminus \{1\} \) is a two-sheeted covering map from \( \mathbb{C} \setminus \{1\} \) to \( \mathbb{C} \setminus \{-1\} \).
   b) Give explicit generators of \( \pi_1(\mathbb{C} \setminus \{1\}, 0) \) and \( \pi_1(\mathbb{C} \setminus \{-1\}, 0) \) and use these to calculate \( P_* \pi_1(\mathbb{C} \setminus \{1\}, 0) \).

(2) Let \( S^1 \) be the unit circle and \( D^2 \) the closed unit disk in \( \mathbb{R}^2 \) whose boundary is \( S^1 \). For each of the following spaces \( X \subset Y \), determine whether there is a retraction \( r : Y \to X \) and give a short justification.
   a) \( X = S^1 \) and \( Y = D^2 \setminus \{(0, 0)\} \), the closed unit disk in \( \mathbb{R}^2 \) with the origin removed.
   b) \( X = S^1 \) and \( Y = D^2 \).

(3) Find all smooth quotients of \( S^4 \) and justify your conclusion.

(4) Suppose \( M \) and \( N \) are connected topological manifolds (not necessarily the same dimension). Recall the wedge product \( M \vee N \) which is the disjoint union of \( M \) and \( N \) with one point in \( M \) identified with one point in \( N \). Use the Mayer-Vietoris sequence to show that \( H_k(M \vee N) = H_k(M) \oplus H_k(N) \) if \( k \neq 0 \). What happens at \( k = 0 \)?

(5) Let \( M \) and \( N \) be topological manifolds of dimension \( m \) and \( n \), respectively. Show that \( m = n \) if \( M \) and \( N \) are homeomorphic.
Part B

(1) a) Prove that the group $SL(2, \mathbb{R})$ of all $2 \times 2$ real matrices of determinant one is a three-dimensional manifold.
   
   b) Give an example of three global vector fields on $SL(2, \mathbb{R})$ that form a basis for the tangent space at the identity matrix.

(2) Let $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ be a vector field in $\mathbb{R}^2$. What condition a differential form $\omega = a(x, y)dx \wedge dy$ must satisfy for the Lie derivative $L_X \omega = 0$? Give an example of a smooth, non-zero two-form $\omega$ in $\mathbb{R}^2 \setminus \{0\}$ such that $L_X \omega = 0$.

(3) Show that on a smooth manifold $M$ there exists a smooth symmetric $(0, 2)$-tensor field $T$ such that $T(v, v) > 0$ for any tangent vector $v \neq 0$.

(4) Let $M$ be a compact, orientable manifold without boundary of dimension $n$. Prove that the $n$-th de Rham cohomology group $H^n_{dR}(M) \neq 0$.

(5) State and prove Stokes theorem.