Manifolds and Topology

Tuesday, 18 April 2017, 1:25-5:25 p.m.

Please use separate bluebooks for parts A and B, and clearly label each bluebook, indicating its contents.

**Most solutions to these problems should be accompanied by proofs.**
Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

This exam is closed-book: no notes or outside assistance is permitted.

All problems carry equal weight.
Part A.

1. Define what it means for two continuous paths $\alpha, \beta : [0,1] \to X$, both starting at the same point $p$ and ending at the same point $q$, to be homotopic. Define the fundamental group $\pi_1(X,x)$ as a set.

2. Suppose that $p$, $q$, and $r$ are points in a space $X$, that $\alpha$ and $\alpha'$ are paths from $p$ to $q$ which are homotopic, and that $\beta$ is a path from $q$ to $r$. Show that the paths $\alpha * \beta$ and $\alpha' * \beta$ from $p$ to $r$ are homotopic.

3. Suppose $X$ is a space. Define what it means for two covering maps $p : Y \to X$ and $q : Z \to X$ to be isomorphic.

4. Suppose $X$ is a (connected, locally contractible) space whose fundamental group is the symmetric group $\Sigma_3$ with six elements. How many isomorphism classes of covering maps $Y \to X$ are there with $Y$ path-connected?

5. Calculate the fundamental group of the space $\mathbb{R}P^2 \vee S^2$, obtained by gluing these two spaces together at a point. Use this to give a geometric description of the universal cover of this space.

6. Describe how a map $\alpha : S^1 \to X$ determines an element in the first homology group $H_1(X)$. Show that two homotopic maps $\alpha$ and $\beta$ give the same element in $H_1(X)$.

7. Suppose $X$ is a space with open subsets $U$ and $V$ such that $X$ is the union $U \cup V$, both $U$ and $V$ are path-connected, and $U \cap V$ is not path-connected (and nonempty). Show that $H_1(X)$ is nontrivial.

8. Suppose $A$ is a subspace of the $n$-disk $D^n$, and that there is a continuous map $r : D^n \to A$ satisfying $r(a) = a$ for all $a \in A$ (a retraction). Show that $H_i(A) = 0$ for all $i > 0$.

9. Define the degree of a continuous map $f : S^n \to S^n$ for $n > 0$.

10. Suppose that $n > 0$ and that a map $f : S^n \to S^n$ is not surjective. Show that $f$ has degree zero.
Part B.

1. Give an example of a smooth function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f'(x) > 0 \) for all \( x \) but where \( f \) is not a diffeomorphism.

2. State the Whitney embedding theorem on embeddings and immersions of \( m \)-dimensional smooth manifolds \( M \) into \( \mathbb{R}^k \).

3. Define \( f(x, y) = (2x^2 + x^4)(y^2 + 1) \), a smooth function from \( \mathbb{R}^2 \) to \( \mathbb{R} \). For this function, determine the: singular points; regular points; singular values; regular values.

4. Suppose \( n < m \), and that \( U \) is an open subset of \( \mathbb{R}^n \). If \( f \) is a smooth function from \( U \) to \( \mathbb{R}^m \), use Sard’s theorem to show that the image of \( f \) cannot contain the ball of radius 1 around the origin.

5. Suppose \((x, y)\) are Cartesian coordinates on \( \mathbb{R}^2 \), and \((u, v)\) are new coordinates given by
   \[
   u = 2x + y + 1 \\
   v = x + y + 3.
   \]
   Express the vector field \( x \frac{\partial}{\partial x} \) in terms of \((u, v)\)-coordinates. Your answer should take the form \( P(u, v) \frac{\partial}{\partial u} + Q(u, v) \frac{\partial}{\partial v} \).

6. Calculate the exterior derivative \( d\omega \), where \( \omega \) is the differential form
   \[xyz \, dx + x^2 e^y \, dy - \cos(x) \, dz.\]
on \( \mathbb{R}^3 \).

7. Let \( M = \{ v \in \mathbb{R}^3 \mid 1 < ||v|| < 72 \} \). For which values of \( k \) does there exist a nonzero \( k \)-form \( \omega \) on \( M \) such that \( d\omega = 0 \), but \( \omega \neq d\tau \) for any \((k-1)\)-form \( \tau \)?

8. Suppose \( X_1, X_2 \) are vector fields on \( \mathbb{R}^3 \) which, at any point, are linearly independent. Define what it means for a surface \( S \subset \mathbb{R}^3 \) to be an \textit{integral surface} for these vector fields, and state the Frobenius theorem relating the existence of integral surfaces for these vector fields to their Lie bracket.

9. Calculate the Lie bracket of the vector fields \( x e^y \frac{\partial}{\partial x} \) and \( y \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) on \( \mathbb{R}^2 \).

10. Suppose that we have a region \( R \) in \( \mathbb{R}^2 \) with a \textit{polygonal boundary}: \((a_i, b_i)\) are points in \( \mathbb{R}^2 \) for \( 0 \leq i \leq n \), \( \gamma_i \) are straight line paths from \((a_i, b_i)\) to \((a_{i+1}, b_{i+1})\), \((a_n, b_n) = (a_0, b_0)\), and that the \( \gamma_i \) go counterclockwise around the boundary of \( R \). Calculate the line integral
    \[
    \int_{\gamma} \frac{1}{2}(xdy - ydx)
    \]
in two ways, both explicitly (getting an answer in terms of \((a_i, b_i)\)) and using Green’s theorem (getting an answer in terms of \( R \)).