Manifolds and Topology Preliminary Exam  
April 22, 2008

No books, papers, or electronic devices may be used in this examination!

**Write only your “codename” on your Blue Books, NOT your actual name!**

This Examination has two Parts, A and B. Solutions of problems in separate Parts must be written in separate Blue Books.

There are four problems in each Part. Each problem has equal weight. The passing score will be based on your total score.

To receive full credit for a problem, the answer must be complete and correct. The scorers must not be expected to supply any missing parts of any answer.

Problems in each Part may be done in any order. You must clearly identify where the parts of your answers are. The scorers will not search at length for answers that are incomplete.

If you use a theorem, state it fully and concisely, or identify it clearly. In either case, verifying hypotheses explicitly is essential.

Do not make assumptions that trivialize a problem.
Part A

1. (a) Let $X$ be a path-connected, locally path connected space. Define a covering space $P : \tilde{X} \rightarrow X$.
   (b) State the theorem which classifies covering spaces of $X$, with fixed base point $x_0 \in X$.
   (c) Assume $\pi(\mathbb{R}P^2, x_0) = \mathbb{Z}/2\mathbb{Z}$ the group with 2 elements ($\mathbb{R}P^2$ is the real projective plane.) How many inequivalent covering spaces are there for $X = \mathbb{R}P^2 \times \mathbb{R}P^2$ (include the trivial ones. Explain)
   (d) Let $P : X \rightarrow X$ be a covering space. A section is a continuous $S : X \rightarrow \tilde{X}$ so that $PS = \text{Id}_x$, the identity. Show that if a section exists, $P$ is a homeomorphism.

2. State (carefully) the Mayer-Vietoris theorem.

3. (a) Define the degree of a map $f : S^n \rightarrow S^n$ (write $\deg(f)$ where $S^n, n > 0$ is the $n$-sphere).
   (b) If the radius of $S^n \subseteq \mathbb{R}^{n+1}$ is 1 and $f, g : S^n \rightarrow S^n$ are two continuous maps with the distance from $f(x)$ to $g(x)$ in $S^n$ less than $\pi$, for all $x$, then $\deg(f) = \deg(g)$
   (c) Show that if $f$ is not onto, $\deg(f) = 0$
   (d) Show that, for any $n > 0$, there is always an onto map of degree 0.

4. (a) Calculate the homology groups of $S^1 \vee S^2 \vee S^3$
   (b) Calculate the homology of the plane, from which $k$ points have been removed.

Part B

1. (a) Define an immersion and an embedding
   
   $f : M^n \rightarrow \mathbb{R}^k$

   (b) If $S^1$ is the circle, show that there is no immersion $f : S^1 \rightarrow \mathbb{R}^1$. More generally, there is no immersion $f : S^n \rightarrow \mathbb{R}^n$
   (c) Explain - in words - why there is an embedding of the Klein bottle in $\mathbb{R}^4$.

2. Consider the following 2-differential form in $\mathbb{R}^3$.
   
   $\omega = f(x, z)dy \wedge dz + g(y, z)dx \wedge dz + dx \wedge dy$

   Here $f(x, z)$ and $g(y, z)$ are smooth functions.
   (a) Calculate $d\omega$.
   (b) Find condition on $f$ and $g$ so that $\omega$ is closed.
   (c) Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be given by $\pi(x, y, z) = (x, y, 0)$. What is $\pi^*(\omega)$?

3. State Sard’s theorem about the regular and critical values of a smooth map $f : M^n \rightarrow \mathbb{R}^k$

4. (a) Show that there is no embedding of a compact 2-manifold $M^2$ in $\mathbb{R}^3$, so that for every point, the Gaussian curvature is negative.
   (b) Describe a non-compact smooth 2-dimensional sub-manifold of $\mathbb{R}^3$, so that at every point the Gaussian curvature is negative.