Manifolds and Topology

Tuesday, August 28, 2018

Please use separate blue books for parts A and B, and clearly label each blue book of its content.

Most solutions to these problems should be accompanied by proofs. Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context and issues involved. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct.

This exam is closed-book: no notes or outside assistance is permitted.

All problems carry equal weight.
Part A

(1) Let $f$ and $g$ be continuous maps from $\mathbb{S}^1$ to $\mathbb{S}^1$. Show that the composition maps $f \circ g$ and $g \circ f$ are homotopic.
(2) Show that $\mathbb{S}^n$ is simply connected for $n > 1$.
(3) Show that $f : \mathbb{S}^n \to \mathbb{S}^n$ has $\deg(f) = (-1)^{n+1}$ if it has no fixed points.
(4) Determine $\pi_1(X)$ if $X$ is a smooth quotient of $\mathbb{S}^{2n}$.
(5) Show that $m = n$ if $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic.
(6) Compute the singular homology groups $H_*(X, \mathbb{Z})$ for $X = \mathbb{S}^2 \times \mathbb{S}^2$.
(7) Let $M^n$ be a smooth compact connected $n$–dimensional submanifold of $\mathbb{R}^m$ with $m \geq n + 2$. Show that $\mathbb{R}^m \setminus M^n$ is connected.
(8) Show that there is no compact three manifold with boundary being the real projective space $\mathbb{R}P^2$. 
Part B

(1) Let \( M = \{(x, y, \sqrt{x^2 + y^2}) \in \mathbb{R}^3; x, y \in \mathbb{R}\} \). Is \( M \) a smooth manifold with the induced topology from \( \mathbb{R}^3 \)? Justify your answer.

(2) Let \( A, B \) and \( C \) be mutually disjoint closed subsets of a smooth manifold \( M \). Show that there exists a smooth function \( f : M \to \mathbb{R} \) such that \( f = 0 \) on \( A \), \( f = \frac{1}{2} \) on \( B \), and \( f = 1 \) on \( C \).

(3) Let \( f \) be a strictly convex smooth function on \( \mathbb{R}^3 \) with minimum value 0. Show that the level set \( \{f = 1\} \) is a smooth manifold and diffeomorphic to \( S^2 \).

(4) Let \( X \) be a smooth vector field on manifold \( M \). Demonstrate the formula

\[
L_X = d i_X + i_X d,
\]

where \( L_X \) is the Lie derivative and \( i_X \) the interior product acting on smooth forms.

(5) Write down explicit formulas for the vector fields \( X \) and \( Y \) which represent the infinitesimal generators of rotations about the \( x- \) and \( y- \) axes respectively and compute their Lie bracket.

(6) Let \( \varphi : S^2 \to T^3 \) be a smooth map. Show that for any top de Rham class \([\nu] \in H^2(T^3)\), we have \( \varphi^* [\nu] = 0 \).

(7) Suppose \( \alpha \) is a closed two-form on \( S^4 \). Show that \( \alpha \wedge \alpha \) must vanish at some point.

(8) Show that for any smooth function \( f(x_1, x_2) \), \( D = \ker(dx_3 - x_1 dx_2) \cap \ker(dx_1 - f(x_1, x_2) dx_2) \subset T\mathbb{R}^4 \) is an integrable distribution.