This exam has two parts. A1, A2, A3, and A4 and B1, B2, B3 and B4. Please use separate blue books for each part A and B. Be sure to put your code name on each book and indicate clearly which problems are in each book. Do not write your real name on any book. Please explain your work clearly and indicate clearly what results you are using in your explanation.

A1. a) An inclusion (continuous) \( i : A \rightarrow X \) is a retraction if there is a continuous \( r : X \rightarrow A \) so that \( ri = Id_A \) (i.e. \( r(i(a)) = a \) for all \( a \in A \)). Which of the following are retractions?

   (i) \( S^1 \subseteq D^2 \) where \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \) and \( D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \).

   (ii) \( i : X \rightarrow X \times X \) where \( i(x) = (x, x) \).

   (iii) Fix \( x_0 \in X \). \( i : X \rightarrow X \times X \) where \( i(x) = (x, x_0) \).

   (iv) \( X \vee X = \{(x, y) \mid \text{either } x = x_0, \text{ or } y = x_0\} \)

   \( i : X \rightarrow X \vee X \) where \( i(x) = (x, x_0) \).

   b) Let \( H(i) : H_n(A) \rightarrow H_n(X) \) be the induced map on homology, \( n \geq 0 \). If \( i \) is a retraction, show that \( H(i) \) is a \( 1-1 \) map of \( H_n(A) \) onto a direct summand of \( H_n(X) \).

A2. a) State carefully the theorem relating the subgroups of \( \pi_1(X, x_0) \) to the covering spaces of \( X \).

   b) Let \( \mathbb{R}P^2 \) be the real projective plane. How many 2-sheeted covering spaces are there of \( X = \mathbb{R}P^2 \times \mathbb{R}P^2 \times \mathbb{R}P^2 \)?

A3. Let \( S^n \) be the n-sphere, \( n > 0 \), \( f : S^n \rightarrow S^n \) a continuous map

   a) Define \( \text{deg}(f) \)

   b) Let \( f, g : S^n \rightarrow S^n \) be 2 continuous maps so that the angle between the vector \( f(x) \) and \( x \) is always \( > 0 \). What is \( \text{deg}(f) \)?

   c) Suppose \( f : S^n \rightarrow S^n \) is not onto. What is \( \text{deg}(f) \)?

A4. Let \( T = S^1 \times S^1 \) = torus.

   a) What are the homology groups \( H_1(T) \) and \( H_2(T) \)?

   b) If \( t : T \rightarrow T \) is given by \( t(a, b) = (b, a) \) what is the effect of \( t \) on \( H_1(T) \) and \( H_2(T) \)?

   c) Describe a continuous \( f : T \rightarrow T \) which is homotopic to \( \text{Id}: T \rightarrow T \), yet has no fixed points.
B1. a) Let $M^n$ be a smooth $n$-manifold. Describe the tangent space $TM^n$.

b) Show that $TS^1 \equiv S^1 \times \mathbb{R}$ (i.e. the spaces are homeomorphic).

c) Show that the space of unit length tangent vectors to $S^2$ is homeomorphic to the $3 \times 3$ orthogonal matrices of determinant 1.

B2. State Stokes' Theorem about the integral of forms over a manifold with boundary.

B3. Let $i : T \rightarrow \mathbb{R}^3$ be a smooth embedding of a torus in $\mathbb{R}^3$.

a) What is the integral of the Gauss curvature of $i(T)$ over all of $i(T)$.

b) Show that $i(T)$ must have a point of negative Gauss curvature.

B4. Consider the following vector fields on $\mathbb{R}^3$.

$$L = y^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + z^2 \frac{\partial}{\partial z}$$

$$M = -\frac{\partial}{\partial x} + z^2 \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Is there a 2-dimensional submanifold containing $(0,0,0)$ whose tangent space is spanned by $L$ and $M$? Explain.