Complex Analysis Preliminary Exam – Fall 2013

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam. All problems have equal weight. Provide complete explanations for your answers. Any standard theorems used in your arguments should be clearly stated.

1. Find the number of the zeros of the polynomial $z^4 + 3z^2 + z + 1$ inside the unit disk.

2. Let $n \geq 2$ be an integer. Evaluate the following integral

$$\int_0^\infty \frac{1}{1 + x^n} \, dx$$

Carefully justify all your steps.

3. Suppose $f$ is analytic on $\{z : 0 < |z| < 1\}$ and $|f(z)| \leq \log\left(\frac{1}{|z|}\right)$. Show that $f$ identically 0.

4. Show that the equation $\sin(z) = z$ has infinitely many solutions in the complex plane.

5. (a) State Schwarz’s Lemma.

   (b) Let $f : D \to D$ be a holomorphic map of the unit disc into itself. Prove that for all $z \in D$, $\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$.

6. Determine all continuous functions on $\{z : 0 < |z| \leq 1\}$ which are harmonic on $\{z : 0 < |z| < 1\}$ and which are identically 0 on $\{z : |z| = 1\}$.

7. (a) Prove that the series $\sum_{n=-\infty}^{n=\infty} \frac{1}{(z - n)^2}$ converges to a meromorphic function on $\mathbb{C}$.

   (b) Prove that there is an entire function $h(z)$ so that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{n=\infty} \frac{1}{(z - n)^2} + h(z)$.

8. Show that the total number of poles of an elliptic function $f$ in its fundamental parallelogram is $\geq 2$. 