Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam. All problems have equal weight. Provide complete explanations for your answers. Any standard theorems used in your arguments should be clearly stated.

1. Show that the polynomial $z^5 + 3z^3 + 7$ has all its zeros in the disk $|z| < 2$.

2. Let $h$ be a nowhere zero, entire holomorphic function. Prove that there exist an entire holomorphic function $g$ such that $e^g = h$.

3. Evaluate the following improper integral:

$$
\int_0^\infty \frac{1}{1 + x^6} \, dx
$$

Carefully justify all your steps.

4. Let $f$ be holomorphic function in the right half-plane $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Suppose that $|f(z)| < 1$ for all $z$ in the domain of $f$, and $f(1) = 0$. Find the largest possible value of $|f(2)|$.

5. Let $f(z)$ be an entire holomorphic function. Suppose that $f(z) = f(z + 1)$ and $|f(z)| \leq e^{\|z\|}$ for all $z \in \mathbb{C}$. Prove that $f(z)$ must be constant.

6. Show that the Möbius transformation maps a straight line or circle onto a straight line or circle.

7. Prove that there is no one-to-one conformal map of the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

8. (a) State Cauchy-Goursat’s Theorem.

   (b) Use Cauchy-Goursat’s Theorem to prove that if the function $f$ is continuous on $\mathbb{C}$ and analytic on every point not on the real axis, then $f$ is analytic everywhere.