Algebra Prelim Written Exam Spring 2015

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] The commutator subgroup $C$ of a group $G$ is the subgroup generated by all commutators $ghg^{-1}h^{-1}$. Show that $C$ is normal, that $G/C$ is abelian, and that any homomorphism $G \to A$ of $G$ to an abelian group $A$ has kernel containing $C$.

[2] Show that a group $G$ of order $pq^2$, with $p$ and $q$ distinct primes, has a proper normal subgroup.

[3] Express $x_1^3 + \ldots + x_n^3$ in terms of the elementary symmetric polynomials in $x_1, \ldots, x_n$.

[4] Show that a finite abelian group of linear operators on a finite-dimensional complex vector space has simultaneous eigenvectors forming a basis.

[5] Show that the ideal generated by 13 and $x^3 - 2$ in $\mathbb{Z}[x]$ is maximal.

[6] Prove that the tenth cyclotomic polynomial

$$\Phi_{10}(x) = \frac{(x^{10} - 1)(x - 1)}{(x^5 - 1)(x^2 - 1)} = x^4 - x^3 + x^2 - x + 1$$

is irreducible in $\mathbb{F}_3[x]$, where $\mathbb{F}_3$ is the finite field with 3 elements.