Algebra Prelim Written Exam Spring 2012

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding the context and of which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent, legible writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, tablets, phones, wireless, bluetooth, or other communication or storage devices may be used during the exam.

[1] Show that all groups of order 35 are cyclic.

[2] Let $G$ be a finite group and $H$ a subgroup of index 2. Show that $H$ is normal.

[3] Let $S, T$ be diagonalizable linear operators on a finite-dimensional $\mathbb{C}$-vectorspace $V$, with $ST = TS$. Show that $V$ has a basis of simultaneous eigenvectors for $S, T$.

[4] Prove that $x^5 + y^7 + z^{11}$ is irreducible in $\mathbb{C}[x, y, z]$.

[5] Describe all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_{12})$, where $\zeta_{12}$ is a primitive twelfth root of unity.

[6] Prove that $x^4 + 1$ is reducible in the polynomial ring $\mathbb{F}_p[x]$ over the finite field $\mathbb{F}_p$ with $p$ elements, for every prime $p$.

[7] Grant that the ring $\mathbb{Z}[i]$ of Gaussian integers is Euclidean, thus is a principal ideal domain. Observe that $(2 + i)(2 - i) = 5$. How many isomorphism classes of $\mathbb{Z}[i]$-modules with exactly 5 elements are there?