Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your **codename**, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Let $p$ be the smallest prime dividing the order of a finite group $G$. Show that a subgroup $H$ of index $p$ is necessarily *normal*.

[2] Classify groups of order $2p$, where $p$ is a prime and $p \neq 2$.

[3] Show that $x^{25} - 10$ factors into linear factors in $\mathbb{F}_{101}[x]$, where $\mathbb{F}_{101}$ is a field with 101 elements.

[4] Let $a, b, c$ be integers with greatest common divisor 1. Show that there is a 3-by-3 integer matrix with determinant 1 having $a, b, c$ as its top row.

[5] Let $R$ be a commutative ring of endomorphisms of a finite-dimensional complex vector space $V$. Prove that there is at least one (non-zero) common eigenvector for $R$ on $V$.

[6] Describe all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

[7] Let $k$ be a field, and $a, b, c, d$ indeterminates. Let $K = k(a, b, c, d)$. Prove that $x^4 + ax^3 + bx^2 + cx + d$ is irreducible in $K[x]$. 
