Algebra Prelim Written Exam Fall 2016

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding of the context and of which issues are primary. Do not make assumptions or choose contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.

[1] Show that every group of order $pq$ with primes $p < q$ and $q \neq 1 \mod p$ is abelian.

[2] Determine the cardinality of the group $GL_3(\mathbb{F}_q)$ of invertible 3-by-3 matrices with entries in the finite field $\mathbb{F}_q$ with $q$ elements.


[5] Show that the ideal $I$ in $\mathbb{Z}[x]$ generated by 13 and $x^2 + 1$ is not maximal.

[6] Let $A$ be an abelian group of unitary operators on a finite-dimensional complex vector space $V$ with hermitian inner product $\langle , \rangle$. Show that there is a orthonormal basis for $V$ consisting simultaneous eigenvectors for all $T \in A$.

[7] Show that $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$, but reducible in $\mathbb{F}_p[x]$ for every prime $p$.

[8] Explicitly determine all fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$, where $\zeta$ is a primitive $8^{th}$ root of unity.