Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration of understanding the context and of which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent, legible writing is essential: your paper should not be a puzzle for the grader.

Write your codename, not actual name, on each booklet. No notes, books, calculators, computers, tablets, phones, wireless, bluetooth, or other communication or storage devices may be used during the exam.

[1] Determine all abelian groups of order 100.

[2] Determine all (not-necessarily-abelian) groups of order 175.

[3] Let $A$ be a finite abelian group of linear operators on a finite-dimensional complex vector space $V$. Show that there is a basis of simultaneous eigenvectors.

[4] Prove that $x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{F}_{13}[x]$, where $\mathbb{F}_{13}$ is the finite field with 13 elements.

[5] Describe all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta_8)$, where $\zeta_8$ is a primitive eighth root of unity.

[6] Show that $x^7 + y^7 + z^7$ is irreducible in $\mathbb{F}_5[x, y, z]$, where $\mathbb{F}_5$ is a field with 5 elements.

[7] Let $T$ be an $n$-by-$n$ matrix with integer entries. Show that there exist integer matrices $A, B$ with determinants $\pm 1$ such that $ATB$ is diagonal.