Questions are equally weighted. Give the essential explanations and justifications: a large part of each question is determination of the crucial points. Do not make assumptions which trivialize the problems. **Put part I solutions in blue books separate from part II solutions.** Write your codename on your exam, NOT your name.

**Part I**

I.1 Let $S, T$ be linear transformations of a finite dimensional vector space $V$ over an algebraically closed field $k$. Suppose that $ST = TS$. Show that $S, T$ have a simultaneous eigenvector, that is, a non-zero vector $v$ such that $Sv = \lambda v$ and $Tv = \mu v$ for $\lambda, \mu$ in $k$.

I.2 Let $\zeta$ be a primitive $35^{\text{th}}$ root of unity in $\mathbb{C}$. Determine all the intermediate fields between the rationals $\mathbb{Q}$ and the extension field $\mathbb{Q}(\zeta)$, and describe the Galois groups.

I.3 Let $k$ be a field $\alpha_1, \ldots, \alpha_n$ distinct elements of $k$. Let $Q(x) \in k[x]$ have degree $< n$. Show that there are unique $c_1, \ldots, c_n \in k$ so that

$$
\frac{Q(x)}{(x - \alpha_1) \ldots (x - \alpha_n)} = \frac{c_1}{x - \alpha_1} + \cdots + \frac{c_n}{x - \alpha_n}
$$

I.4 Show that $x^6 + x^3 + 1$ factors into two irreducible cubics over the 49-element field $\mathbb{F}_{49}$.

I.5 Show that $x^5 + y^5 + z^5$ is irreducible as a polynomial in $\mathbb{C}[x, y, z]$.

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**Part II**

II.1 Let $f(x) \in \mathbb{Q}[x]$ be of prime degree $p$, and irreducible. Assume that $f(x) = 0$ has $p - 2$ real roots and 2 complex roots. Prove that the splitting field of $f$ over $\mathbb{Q}$ has Galois group the full symmetric group on $p$ letters.

II.2 Let $k$ be a field and $p$ a prime number. Let $a \in k$ such that $a$ is not a $p^{\text{th}}$ power. Show that $x^p - a$ is irreducible in $k[x]$.

II.3 Let $G$ be a group of order $p^2q$ where $p$ and $q$ are distinct primes. Show that $G$ is not simple (that is, show that $G$ has a non-trivial normal subgroup).

II.4 Let $k$ be a field, $f(x) \in k[x]$ a quartic with 4 distinct roots $\mu_1, \mu_2, \mu_3, \mu_4$ in an algebraic closure $\overline{k}$ of $k$. Let $F = k(\mu_1, \mu_2, \mu_3, \mu_4)$. Let

$$
\alpha = \mu_1\mu_2 + \mu_3\mu_4 \quad \beta = \mu_1\mu_3 + \mu_2\mu_4 \quad \gamma = \mu_1\mu_4 + \mu_2\mu_3
$$

Let $E = k(\alpha, \beta, \gamma)$. Show that $\text{Gal}(E/k) \cong G/(G \cap H)$ where $G = \text{Gal}(F/k)$, and $H = \{1, (1 2)(3 4), (1 3)(2 4), (1 4)(2 3)\} \subset S_4$ where we identify $G$ with a subgroup of $S_4$ by its permutation action on $\mu_1, \mu_2, \mu_3, \mu_4$.

II.5 Let $R$ be a commutative ring with 1. Let $I, J$ be ideals in $R$. Let

$$(I : J) = \{r \in R : rJ \subset I\}$$

Recall: an ideal $U$ is primary if $xy \in U$ and $x \not\in U$ implies $y^n \in U$ for some $n > 0$. And recall that the radical $\sqrt{U}$ is $\{r \in R : r^n \in U$ for some positive integer $n\}$. Prove that if $I$ is primary and if $J$ is not a subset of $I$ then $(I : J)$ is primary, and, further, that $\sqrt{(I : J)} = \sqrt{I}$.