The Self-financing Condition: Remembering the Limit Order Book

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Standard Assumptions in Finance

Black-Scholes theory

- Price given by a single number
- infinite liquidity
  - one can buy or sell any quantity at this price
  - with NO IMPACT on the asset price
- Fixes to account for liquidity frictions
  - Introduction of Transaction Costs
  - Add Liquidity constraints ∼ transaction costs

Not satisfactory for

- Large trades (over short periods)
- High Frequency Trading (HFT)

Need Market Microstructure

- e.g. understand how are buy and sell orders executed?
New Markets

▶ Quote Driven Markets
  ▶ Market Maker or Dealer centralizes buy and sell orders and provides liquidity by setting bid and ask quotes.
    Ex: NYSE specialist system

▶ Order Driven Markets
  ▶ electronic platforms aggregate all available orders in a Limit Order Book (LOB)
    Ex: NYSE, NASDAQ, LSE
  ▶ Same stock traded on several venues
  ▶ Price discovery made difficult as most instruments can be traded off market without printing the trade to a publicly accessible data source
    ▶ Competition between markets leads to lower fees and smaller tick sizes
  ▶ Creation of Dark Pools
  ▶ Increase in updating frequency of order books
High Frequency Trading

Speculative figures – Sound plausible
► HFT accounts for 60 – 75% of all share volume.
► 10% of that is predatory \( \approx 600 \) million shares per day
► At $0.01-$0.02 per share, predatory HFT is profiting $6-$12 million a day or $1.5-$3 billion a year

Algorithmic Trading – Source of concern
► Moving computing facilities closer to trading platform (latency)
► Relying on / competing with Benchmark Tracking execution algorithms
Pros & Cons

Pros

▶ Smaller tick size;
▶ HFTs provide liquidity
▶ Dark pools reduce trade execution costs from price impact
▶ Markets are more efficient

Cons

▶ Expensive technological arms race
▶ Dark trading incentivizes price manipulation, fishing and predatory trading
▶ Little or no oversight possible by humans (e.g. flash crash) & increased systemic risk
▶ HFT algorithms do not use economic fundamentals (e.g. value & profitability of a firm)
Some Highly Publicized Mishaps

Flash Crash of May 6, 2010

▶ Dow Jones IA plunged about 1000 points (recovered in minutes)
▶ Biggest one-day point decline (998.5 points)
  ▶ At 2:32 pm a mutual fund program started to sell 75,000 E-Mini S&P 500 contracts (≈ 4.1 billion USD) at an execution rate of 9%
  ▶ HFT programs were among the buyers: quickly bought and resold contracts to each other
  ▶ hot-potato volume effect, combined sales drove ”the E-mini price down 3% in just 4 minutes

Other Notable Crashes

▶ Associated Press’ Twitter account hack
  ▶ White House bombed
  ▶ President Obama injured
  ▶ DJIA lost 140 points and recovered in minutes
▶ Several mini flash crashes on NASDAQ in 2012
Some Remarks

- “There is no question that the goal of many HFT strategies is to profit from LFTs’ mistakes. [...] Part of HFT’ success is due to the reluctance of LFT to adopt (or even to recognize) their paradigm.” Maureen O’ Hara

- HFT is not going away

- Speed is important, but not the fundamental difference between HFT and LFT.

- **Microstructure matters!** It drives trades and prices, not fundamentals.

- Key ’tool’ differentiating HFT from LFT: **event-based clock** vs calendar clock.

- Academia, and some LFTs, should learn from incorporating the **HFT paradigm**.
Ambitious Research Program

1. Understand, at the **microscopic** level, **structural relationships** and strategies that HFTs exploit.

2. Identify which microscopic features matter at the **macroscopic** level, and provide **models** on that scale.

3. Use these models to update LFT models and provide **monitoring tools**: transaction cost analysis, measure of toxicity of order flow...
Limit Order Book (LOB): a Crash Course

List of all the waiting **buy** and **sell** orders

- Prices are multiple of the tick size
- For a given price, orders are arranged in a **First-In-First-Out** (FIFO) stack
- At each time $t$
  - The **bid** price $B_t$ is the price of the highest waiting **buy** order
  - The **ask** price $A_t$ is the price of the lowest waiting **sell** order
- The state of the order book is modified by **order book events**:
  - **limit orders**
  - **market orders**
  - **cancelations**

- **consolidated order book**: If the stock is traded in several venues, one aggregates over all (**visible**) trading venues.
- Here, **little** or **no** discussion of **pools**
The Role of a LOB

- Crucial in high frequency finance: explains *transaction costs*.
- **Liquidity providers** post trading intentions: Bids and Offers.
- **Liquidity takers** execute certain orders: adverse selection.

*Figure*: Snapshot of Apple order book at 8:43 (NASDAQ)
Limit Orders

A limit order sits in the order book until it is

▶ either executed against a matching market order
▶ or it is canceled

A limit order

▶ may be executed very quickly if it corresponds to a price near the bid and the ask
▶ may take a long time if
  ▶ the market price moves away from the requested price
  ▶ the requested price is too far from the bid/ask.
▶ can be canceled at any time

Typically, a limit order waits for a match

▶ transaction cost is known
▶ execution time is uncertain
Market Orders

A **market order** is an order to buy/sell a certain quantity of the asset at the **best available price** in the book.

- Agents can put a **market order** that, for a buy (resp. sell) order,
  - the first share(s) will be traded at the ask (resp. bid) price
  - the remaining one(s) will be traded some ticks upper (resp. lower)
  in order to fill the order size.
- The ask (resp. bid) price is then modified accordingly.
- When either the bid or ask queue is **depleted** by
  - market orders
  - cancelations
  the price is **updated** up or down to the next level of the order book.

Typically a **market order** consumes the cheapest limit orders

- **immediate execution** (if the book is filled enough)
- **price** per share instead **uncertain** (depends upon the order size)
Cancellations

- Agents can put a \textit{cancellation} of $x$ orders in a given queue reduces the queue size by $x$

- When either the bid or ask \textit{queue is depleted} by market orders and cancellations, the \textit{price moves} up or down to the next level of the order book.
LOB Dynamics Summary

- Actual trades come in two forms
- Agents can put a **limit order** and wait that this order matches another one
  - transaction cost is known
  - execution time is uncertain
- Agents can put a **market order** that consumes the cheapest limit orders in the book
  - immediate execution (if the book is filled enough)
  - price per share instead depends on the order size
  For a buy (resp. sell) order, the first share will be traded at the ask (resp. bid) price while the last one will be traded some ticks upper (resp. lower) in order to fill the order size. The ask (resp. bid) price is then modified accordingly.
- Agents can put a **cancellation** of $x$ orders in a given queue reduces the queue size by $x$
- When either the bid or ask queue is depleted by market orders and cancelations, the price moves up or down to the next level of the order book.
Market Impact of Large Fills

- Current **mid-price** $p_{mid} = (p_{Bid} + p_{Ask})/2 = 13.98$

- Fill size $N = 76015$ (e.g. buy)

- $n_1$ shares available at best bid $p_1$, $n_2$ shares at price $p_2 > p_1$, \ldots

- $n_k$ shares at price $p_k > p_{k-1}$
  
  $N = n_1 + n_2 + n_3 + \cdots + n_k$

- **Transaction cost**
  
  $n_1 p_1 + n_2 p_2 + \cdots + n_k p_k = 1064578$

- **Effective price**
  
  $p_{eff} = \frac{1}{N}(n_1 p_1 + n_2 p_2 + \cdots + n_k p_k)$

  $= 14.00484$
A LOB Idiosyncrasy: Hidden Liquidity

- Some exchanges (e.g. NASDAQ & NYSE) allow **Hidden Orders**
- Made **visible** to the broader market **after being executed**
- **Controversial**
  - barrier to the implementation of a fully transparent market
  - impediment to price discovery and information dissemination

**Results of First Empirical Analyzes**

- Encourage **fishing**
- After it is **revealed** that a hidden order was executed
  - rash increase of order placement inside the bid-ask after
- HF Traders **divided** in two groups
  - Traders try to take advantage of the remaining hidden liquidity
  - Traders try to steal execution priority from the fully hidden orders
"Partially Hidden" Orders: Iceberg Orders

- **Dark liquidity** posted inside the LOB
- Two components: the **shown quantity** and the **hidden remainder**
- Order queued with the **lit** part of the LOB, only the shown quantity is visible
- When the order reaches the front of the queue, **only the display quantity is filled**
- Trade (price & quantity filled) **revealed**
- **hidden part** put at the back of the queue
- Sometimes **extra execution fee** charged by the exchange
Dark Pools / Crossing Networks

- **Electronic engine** that matches buy and sell orders without routing them to lit exchanges
- **Raison d’être:** move large amounts without impacting the price (no need for iceberg orders)
- **Run by private brokerages**
  - Ex: Liquidnet, Pipeline, ITG’s Posit, Goldman’s SIGMA X.
  - Participants submit (wish) lists of orders to a matching engine
  - Matched orders are executed at the midpoint of the bid-ask spread.
- **Pros:** trade at mid-point can be better than on a lit market
- **Cons:** May have to wait a long time for a match to occur
- Regulated by SEC (in the US) as **Alternative Trading Systems**
  - Little or no public disclosure
  - Not much has been done to increase *transparency*
- Trading on dark pools ≈ 32% of trades in 2012 (!)
Today’s Talk: Search for Structural Relationships

From LOB Models to Low Frequency Models

- Understand transition from discrete to continuous models
- Incorporate market microstructure into low-frequency models.
- Differentiate between liquidity takers and providers
- Identify self-financing conditions

Searching for Answers in the Data

- Nasdaq ITCH data includes all limit and market orders
- Perfect reconstruction of visible limit order book.

Example: KO (Coca Cola) on 18/04/13.
Today’s Talk: Search for Structural Relationships

From LOB Models to Low Frequency Models

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Searching for Answers in the Data

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Midprice

Conventions

- **Trade clock**, \( n = 1, \ldots, N \) corresponds to the times \( t_1 < \ldots < t_N \) at which trades occur.

- **Notation**: \( \Delta_n \chi = \chi_{n+1} - \chi_n \).

- \( \rho_n \): mid-price just before the trade at time \( t_n \) (i.e. \( \rho_{t_n^-} \)).
Goal: All trades happen at the best bid or best ask

- First: remove trades against hidden orders
- Check the result

- Next: remove trades with special deals
Bid-Ask Spread

Convention, Notation, Assumption

- **All** trades (100%) happen at the **best bid** or **best ask**
- $s_n$: bid-ask **spread** just before the trade.
- $s_n \approx |\Delta_n p|$. 

![Price path with spread (zoom in)](image1)

![Price path with spread](image2)
Aggregate inventory

Conventions and comments

- Inventory of the **aggregate liquidity provider**
- $L_n$: **Inventory** just before the trade.
- $\Delta_n L < 0$ means that a **market order** bought at the ask.
Price Impact

Empirical Fact

- $\Delta_n L \Delta_n p \leq 0$ holds 99.1% of the time.
- Prices move in favor of market orders (adverse selection)
Statistical Test of the Hypothesis

Assume mid-price $p$ and inventory $L$ are Itô processes

\[
\begin{aligned}
dp_t &= \mu_t dt + \sigma_t dW_t \\
dL_t &= b_t dt + l_t dW'_t
\end{aligned}
\]

with $d[W, W']_t = \rho_t dt$. Let $p^N$ and $L^N$ be discrete samplings

\[
p^N_n = p_{n/N} \quad \text{and} \quad L^N_n = L_{n/N}
\]

Define

\[
\begin{aligned}
C^N_t &= \sum_{n=1}^{\lfloor Nt \rfloor - 1} \Delta_n p^N \Delta_n L^N \\
V^N_t &= N \sum_{n=1}^{\lfloor Nt \rfloor - 2} \left( \left( \Delta_n p^N \Delta_{n+1} L^N \right)^2 + \Delta_n p^N \Delta_n L^N \Delta_{n+1} p^N \Delta_{n+1} L^N \right)
\end{aligned}
\]

Then

\[
\mathcal{L} \left( \frac{C^N_t - [p, L]_t}{\sqrt{N^{-1} |V^N_t|}} \right) \to N(0, 1)
\]

Confidence intervals for $[W, W']_t$ and test of $H_0: \exists t \in [0, 1], \rho_t > 0$
# Tests Results

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(Integrated) Quadratic Covariations

Rescaled plot of quadratic covariations
Cash Account

Convention and Comment

- $K_n$: **cash** holdings just before the trade.
- **Self-financing** by construction: changes in cash are the amounts exchanged during trades. **No more, no less**
Wealth Definition

\[ X_n = L_n p_n + K_n \]

**Accounting rule**: wealth is the value of the inventory marked to the mid-price plus the cash holdings.
Self-financing equations

Possible wealth dynamics

\[ \Delta_n X = L_n \Delta_n p \]  \hspace{1cm} (1)

\[ \Delta_n X = L_n \Delta_n p + \frac{s_n}{2} |\Delta_n L| \]  \hspace{1cm} (2)

\[ \Delta_n X = L_n \Delta_n p + \frac{s_n}{2} |\Delta_n L| + \Delta_n p \Delta_n L \]  \hspace{1cm} (3)
Self-financing equations

Possible wealth dynamics

\[ \Delta_n X = L_n \Delta_n p \]  
\[ \Delta_n X = L_n \Delta_n p + \frac{s_n}{2} |\Delta_n L| \]  
\[ \Delta_n X = L_n \Delta_n p + \frac{s_n}{2} |\Delta_n L| + \Delta_n p \Delta_n L \]

Corresponding relationships for self-financing cash

\[ \Delta_n K = -p_{n+1} \Delta_n L \]  
\[ \Delta_n K = -p_{n+1} \Delta_n L + \frac{s_n}{2} |\Delta_n L| \]  
\[ \Delta_n K = -p_n \Delta_n L + \frac{s_n}{2} |\Delta_n L| \]
Comparing the three wealth equations

Comments

- **True wealth** coincides with (3).
- Difference between (1) and (2) (**transaction costs**) is large.
- Difference between (2) and (3) (**price impact**) cannot be neglected.
The continuous limit

What are the issues?

- 3 self-financing wealth conditions to choose from;
- **Adverse Selection** constraint;
- Choice of assumptions on $p$ and $L$: jumps? finite variation?
- Bid-ask spread: fixed? Vanishing?

Informally

want (3) to include transaction costs and price impact

$p_{N}^{n} = p_{\lfloor n/N \rfloor}$ and $L_{N}^{n} = L_{\lfloor n/N \rfloor}$ from $p_{t}$ and $L_{t}$ continuous Itô processes sampled at $1/N$, $2/N$, ..., $1$ (trade clock)

so $\Delta n^{p_{N}} = O(1/\sqrt{N})$ and $\Delta n^{L_{N}} = O(1/\sqrt{N})$ as $N \to \infty$

We will also want $s_{N}^{n} = O(1/\sqrt{N})$
The continuous limit

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- We will also want $s^N_n = O(1/\sqrt{N})$
Continuous setup

Continuous data

Assume

\[
\begin{align*}
dp_t &= \mu_t dt + \sigma_t dW_t \\
\mathrm{d}L_t &= b_t \mathrm{d}t + \ell_t \mathrm{d}W'_t
\end{align*}
\]

for some \( \mu_t, b_t, \sigma_t > 0, \ell_t > 0 \), adapted, and

\[
[W, W']_t = \int_0^t \rho_s ds \quad \text{for some } \rho_t \in [-1, 1]
\]

and assume \( s_t \) is continuous and adapted
Continuous setup

Continuous data
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\[ \begin{align*}
  dp_t &= \mu_t \, dt + \sigma_t \, dW_t \\
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\end{align*} \]
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\[ [W, W']_t = \int_0^t \rho_s \, ds \quad \text{for some } \rho_t \in [-1, 1] \]

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Discretization choice

\[ p^N_n = p_{[n/N]}; \quad L^N_n = L_{[n/N]} \]

and

\[ s^N_n = \frac{1}{\sqrt{N}} s_{[n/N]} \]
Proposed discrete equations

Wealth dynamics

\[ \Delta_n X^N = L_n^N \Delta_n p^N + \frac{S_{\lfloor n/N \rfloor}}{2\sqrt{N}} |\Delta_n L^N| + \Delta_n p^N \Delta_n L^N \]

Price impact constraint

\[ \Delta_n p^N \Delta_n L^N \leq 0 \]

If we want the discretization to mimic the micro structure of a LOB
Back to the diffusion limit

Wealth dynamics

\[ dX_t = L_t dp_t + \frac{s_t \ell_t}{\sqrt{2\pi}} dt + d[L, p]_t \]

Price impact constraint

A necessary condition for an inventory obtained by limit orders is:

\[ d[L, p]_t \leq 0 \]
Applications: I. Hedging

Model assumptions

Price model:
\[ dp_t = \mu(t, p_t) dt + \sigma(t, p_t) dW_t \]

Inventory model:
\[ dL_t = b_t dt - \ell_t dW_t \]

Spread model: (empirical studies)
\[ s_t = \sqrt{2\pi\lambda} \sigma_t, \quad \text{with} \quad \lambda > 1/2 \]

No dividend or interest rate.

NB: Note that \( \lambda = 1 \) implies \( dX_t = L_t dp_t \) (frictionless case)
Applications: 1. Hedging

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Objective

Given the model for \( p \) and \( s \), find \( L \) such that \( X \) hedges a European option with payoff \( f(p_T) \).
Replication argument

Markovian setup, so price of the option given by function \( \nu(t, p) \).

Itô’s formula:

\[
d(X_t - \nu(t, p_t)) = (L_t - \Delta_t) dp_t - (\Theta_t + \frac{1}{2} \Gamma_t \sigma^2(t, p_t)) dt
\]

\[
+ \frac{s_t}{\sqrt{2\pi}} \ell_t dt + d[p, L]_t
\]
**Replication argument**

Markovian setup, so price of the option given by function $v(t, p)$. Itô’s formula:

$$d(X_t - v(t, p_t)) = (L_t - \Delta_t) dp_t - (\Theta_t + \frac{1}{2} \Gamma_t \sigma^2(t, p_t)) dt$$

$$+ \frac{s_t}{\sqrt{2\pi}} \ell_t dt + d[p, L]_t$$

**Matching Itô decompositions**

$$L_t = \Delta_t$$

which also implies

$$-\ell_t = \Gamma_t \sigma(t, p_t)$$
Final Solution

Delta hedging

\[
\begin{align*}
L_t &= \Delta_t \\
\ell_t &= -\Gamma_t \sigma(t, p_t)
\end{align*}
\]

Only negative Gamma options can be replicated via limit orders!
Final Solution

Delta hedging

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\begin{aligned}
L_t &= \Delta_t \\
\ell_t &= -\Gamma_t \sigma(t, p_t)
\end{aligned}
\]

Only negative Gamma options can be replicated via limit orders!

Pricing PDE

\[
\partial_t v(t, p) + \left( \lambda - \frac{1}{2} \right) \sigma^2(t, p) \partial_p^2 v(t, p) = 0
\]

local volatility multiplied by a factor of $\sqrt{2\lambda - 1}$
Applications. II Market making

Setting

Still, aggregate market maker.

- Price $p_t$ exogenously given
- Sole control: bid-ask spread $s_t$.
- Affects inventory $L_t$.
- Price impact included through correlation between inventory and price.
Objectives

Mathematical Problem
Solve the optimal control problem of a risk-neutral representative market maker.

Model Insights
- What macro-quantities the market maker is long.
- What factors affect the optimal bid-ask spread.
Microscopic model

Modified **Almgren & Chriss** model

\[ \Delta_n L = -\lambda_{n+1} \Delta_n \rho \]

Modified **Avellaneda & Stoikov** model

\[ \mathbb{E}[\lambda_{n+1} | \mathcal{F}_n] = \rho_n(s_n)f_n(s_n) \]
\[ \mathbb{E}[\lambda_{n+1}^2 | \mathcal{F}_n] = f_n(s_n)^2 \]
Macroscopic model

Inventory
Recall, $p_t$ given exogenously,

$$dL_t = -\rho_t(s_t)f_t(s_t)dp_t + f_t(s_t)\sqrt{1 - \rho_t^2(s_t)\sigma_t}dW_t$$

First term is standard linear price impact. Second term is the non-toxic order flow (in the terminology of O’ Hara et.al).

Objective function

$$\mathbb{E}X_T$$

(risk neutral market maker)
(Pontryagin) stochastic maximum principle

Solution can be reduced via martingale methods to finding the maximum of the function:

$$F_t : s \mapsto \frac{s}{\sqrt{2\pi}\sigma_t} f_t(s) - \alpha_t \rho_t(s) f_t(s)$$

where

$$\alpha_t = \mathbb{E} [p_T - p_t | \mathcal{F}_t] \frac{\mu_t}{\sigma_t^2} + \frac{Z_t}{\sigma_t}$$

with $Z_t$ the volatility of $\mathbb{E} [p_T | \mathcal{F}_t]$. 
Extra assumption

Homogenization

Assume $f_t$ and $\rho_t$ to be of the form

$$\rho_t(s) = \rho(t/\sigma_t); \quad f_t(s) = f(s_t/\sigma_t)$$

Consequence

Optimal spread:

$$s_t^* = \sigma_t m(\alpha_t)$$

P&L:

$$\mathbb{E}X_T = \mathbb{E} \left[ \int_0^T M(\alpha_t) \sigma_t^2 dt \right]$$

for some functions $m$ and $M$. $M$ is always decreasing. $m$ is increasing under certain uniqueness assumptions.
Special case 1: martingale market

**Price model**

If

$$dp_t = \sigma_t dW_t$$

then

$$\alpha_t = 1$$

**Consequence**

**Benchmark** case. *Linear relationship* between spread and volatility.
Market maker is long volatility as long as he is profitable ($M(1) > 0$).
**Special case 2: momentum market**

**Price model**

\[ dp_t = \mu p_t dt + \sigma p_t dW_t \]

GBM (Samuelson) with \( \mu > 0 \), then

\[ \alpha_t = \frac{\mu}{\sigma^2} \left( e^{\mu(T-t)} - 1 \right) + e^{\mu(T-t)} \]

**Consequence**

\( \alpha_t > 1 \), \( \alpha_t \) is a deterministic, decreasing function of \( t \).

The market maker quotes **larger spreads**, expects **less profit** and captures less volume in the ’momentum’ Black-Scholes model.
Special case 3: mean-reverting market

Price model

\[ dp_t = -\rho (p_t - p_0) dt + \sigma dW_t \]

mean reverting OU with \( \rho > 0 \), then

\[ \alpha_t = -\frac{\rho}{\sigma^2} (p_t - p_0)^2 \left( e^{-\rho(T-t)} - 1 \right) + e^{-\rho(T-t)} \]

Consequence

\( \alpha_t < 1 \) iff \( (p_t - p_0)^2 < \frac{\sigma^2}{\rho} \)

Unless the price is significantly away from its long-term trend, the market maker quotes smaller spreads, expects more profit and captures more volume in the 'mean reverting' Ornstein-Uhlenbeck model.
Summary of the three testbed cases

**Martingale market**

\[ \frac{s_t}{\sigma_t} \] is a constant. Market maker is on average long the integrated volatility.

**Momentum market**

\[ \frac{s_t}{\sigma_t} \] is an increasing function of \( T - t \). Profits are smaller and spreads larger than in the martingale market.

**Mean-reverting market**

\[ \frac{s_t}{\sigma_t} \] is an increasing function of \( (p_t - p_0)^2 \). Profits are typically larger and spreads typically smaller than in the martingale case.
Toxicity index

Motivation
O’Hara defines a toxicity index as a measure of the adverse selection limit orders are subject to. Useful for:

▶ Deciding whether to use limit or market orders.
▶ Market making.
▶ Understanding flash crashes.

Different interpretations
O’Hara’s toxicity index is based on an informed trader model and only looks at trade volumes. We propose within our price impact framework, an index which takes into account both trade volumes and price changes.
Two forms of Toxicity

**Instantaneous toxicity**

\[ \rho_t = -\text{corr}(\Delta p, \Delta L)_t \]

is an estimate of the instantaneous correlation between price and inventory variations. It represents the proportion of incoming market orders that move the price.

**Integrated toxicity**

\[ r = -2 \frac{\sum \Delta_n p \Delta_n L}{\sum s_n |\Delta_n L|} \]

measures the ratio between the money lost to price impact, and the money collected through spread. De facto, Market Makers hold an option on this ratio.
<table>
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<th>Stock</th>
<th>correlation</th>
<th>r_toxicity</th>
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<td>AAPL</td>
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▶ We propose an equation which takes into account:
  ▶ transaction costs
  ▶ price impact
  ▶ differentiates between limit orders and market orders

▶ Same level of complexity
▶ Similar equation for market orders
▶ Fits data very well (tested on a pool of 120 stocks selected for an ECB study of HFT).
▶ Generalizes to a full LOB
▶ Needs more attention

▶ Relate $\rho_t \leq 0$ to the queuing systems in LOBs
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