Spin chains of Haldane–Shastry type: an overview

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Joint work with J.C. Barba, B. Basu-Mallick, A. Enciso, F. Finkel, M.A. Rodríguez
Some references

Outline of the talk

1 Quantum Calogero–Sutherland models
   - Scalar models
   - Spin CS models

2 Spin chains of Haldane–Shastry type
   - The HS chain
   - The freezing trick
   - The partition function

3 Quantum chaos
   - Basic concepts
   - Level density
   - Distribution of the spacings

4 Conclusions
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Scalar models of Calogero and Sutherland

- **Calogero** model (1971): \( N \) quantum particles in 1-D, with Hamiltonian

\[
H = -\nabla^2 + \omega^2 r^2 + a(a - 1) \sum_{1 \leq i \neq j \leq N} v(x_i - x_j)
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\((\omega > 0, a > \frac{1}{2})\) and interaction potential

\[v(x) = x^{-2}\]
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- **Sutherland** model (1971): $\omega = 0$,

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Scalar models of Calogero and Sutherland

- The above models are associated to the root system

\[ A_{N-1} \equiv \{x_i - x_j \mid 1 \leq i \neq j \leq N\} \subset (\mathbb{R}^N)^* \]

Olshanetsky & Perelomov [PR 81] showed that one can construct CS integrable models associated to any (extended) root system

- Example:

\[ B_C = \{x_i \pm x_j \mid 1 \leq i \neq j \leq N\} \cup \{x_i, 2x_i \mid 1 \leq i \leq N\} \]

\[ H = -\Delta + \sum_{i \neq j} [v(x_i - x_j) + v(x_i + x_j)] + \sum_{i} [v(x_i) + v(2x_i)] \]
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Scalar models of Calogero and Sutherland

Properties

- These models are **completely integrable**: there exist \( N \) self-adjoint (differential) operators \( I_1 \equiv H, \ldots, I_N \) such that \([I_j, I_k] = 0, \forall j, k\).

- They are also **exactly solvable**: for instance, for the Calogero model

\[
\begin{align*}
E_n &= 2\omega \sum_i n_i + \omega N (a(N - 1) + 1), \\
\psi_n(x) &= \mu(x) P_n(x),
\end{align*}
\]

with \( n_1 \geq n_2 \geq \ldots \geq n_N \geq 0, n_i \in \mathbb{N}, \) and

\[
\begin{align*}
\mu(x) &= e^{-\frac{\omega}{2} r^2} \prod_{i<j} |x_i - x_j|^a \equiv \text{ground state} \\
P_n(x) &= \text{polynomial in } x \ (\text{related to Jack polynomial})
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Spin CS models
For each of the scalar CS models, it is possible to construct an associated integrable and exactly solvable $su(m)$ spin CS model.
Spin CS models

- **Internal space**: $\mathbb{C}^m \otimes \cdots \otimes \mathbb{C}^m\{N\}$

- **Spin basis**: 
  \[
  \left\{ \left| s_1 \right\rangle \otimes \cdots \otimes \left| s_N \right\rangle \right\} \equiv \left| s_1, \ldots, s_N \right\rangle,
\]
  
  \[s_k \in \{1, \ldots, m\} \equiv \text{spin of } k\text{-th particle}\]

- **Spin permutation operators**: 
  \[
  S_{ij} \left| s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N \right\rangle = \left| s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N \right\rangle
  \]

- The permutation operators $S_{ij}$ can be expressed in terms of the standard $su(m)$ generators $J_k \equiv (J_k^1, \ldots, J_k^{m^2-1})$ as
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  S_{ij} = 2 J_i \cdot J_j + \frac{1}{m}
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- **Spin Calogero model** [Minahan & Polychronakos PLB 93]:

$$H = -\Delta + \omega^2 r^2 + \sum_{i \neq j} \frac{a(a \pm S_{ij})}{(x_i - x_j)^2}$$

- **Spin Sutherland model** [Ha & Haldane PRB 92]:

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The Haldane–Shastry chain
A spin chain is a one-dimensional lattice whose sites interact through their spins.
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- Haldane–Shastry spin chain [PRL 88]:

\[ \mathcal{H} = \frac{1}{2} \sum_{i<j} \sin^{-2}(\varphi_i - \varphi_j)(1 \pm S_{ij}), \]
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\[2\vartheta_i\]

\[\Rightarrow \text{ } N \text{ equally spaced spins on a circle}\]
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Motivation: limiting case of the one-dimensional Hubbard model, relevant for high-temperature superconductivity
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**Properties**

- Long-range exchange interactions whose strength depends on the spins’ positions
- The model is completely integrable [Fowler & Minahan PRL 93] and exactly solvable [Bernard et al. JPA 93]
- The spectrum, highly degenerate due to the symmetry of \( \mathcal{H} \) under the Yangian algebra \( \mathcal{Y}(\mathfrak{gl}_m) \), is a (complicated) subset of the integers
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Polychronakos’s freezing trick
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Goal:

Compute the partition function

\[ Z(T) = \sum_n q^{E_n}, \quad q \equiv e^{-\frac{1}{k_B}T} \quad (T \equiv \text{temperature}), \]

of the Haldane–Shastry spin chain by exploiting its connection with the scalar and spin Sutherland models.
Polychronakos’s freezing trick

- The Hamiltonian $\mathcal{H}$ of the HS chain is related to those of the spin and scalar Sutherland models, $H$ and $H^{sc}$, by

$$\mathcal{H} = h(\vartheta), \quad H = H^{sc} + 4a \ h(x), \quad h(x) \equiv \sum_{i<j} \frac{1 \pm S_{ij}}{2 \sin^2(x_i - x_j)}$$
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In the strong coupling limit $a \to \infty$,

$$H = -\Delta + a^2 U(x) + O(a), \quad U(x) \equiv \sum_{i \neq j} \sin^{-2}(x_i - x_j)$$
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$\implies$ as $a \to \infty$, the particles concentrate around the (unique, up to global translations) minimum of the potential $U$ in the $A_{N-1}$ Weyl alcove $C \equiv \{x | x_1 < x_2 < \cdots < x_N < x_1 + \pi\}$
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- Key fact: the coordinates of the minimum of $U$ in $C$ are precisely the sites $(\vartheta_1, \ldots, \vartheta_N)$ of the HS chain, so that

$$H \underset{a \to \infty}{\simeq} H^{sc} + 4a \, h(\vartheta)$$
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$$H \underset{a \to \infty}{\simeq} H^{sc} + 4a \mathcal{H} \quad \implies \quad E_{ij} \underset{a \to \infty}{\simeq} E_{ij}^{sc} + 4a \mathcal{E}_j,$$

where $E_{ij}, E_{ij}^{sc}, \mathcal{E}_j \equiv$ eigenvalues of $H, H^{sc}, \mathcal{H}$
Polychronakos’s freezing trick

If $Z, Z^{sc}, \mathcal{Z} \equiv$ partition functions of $H, H^{sc}, \mathcal{H}$, then

$$Z(T) \sim_{a \to \infty} Z^{sc}(T) \mathcal{Z}(T/(4a))$$
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A similar analysis is valid for the other CS models. For instance, the spin Calogero model yields the Polychronakos–Frahm chain

$$\mathcal{H} = \sum_{i < j} \frac{1 \pm S_{ij}}{(\xi_i - \xi_j)^2},$$

where $\xi_k = k$-th zero of the Hermite polynomial $H_N$
The partition function of the HS chain
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Finkel & AGL PRB 2005

The partition function of the (antiferromagnetic) Haldane–Shastry $su(m)$ spin chain is given by

$$Z(T) = \sum_{k \in \mathcal{P}_N} q^{\sum_{i=1}^{r-1} \epsilon(K_i)} \prod_{i=1}^{r} \binom{m}{k_i} \cdot \prod_{i=1}^{N-r} (1 - q^{\epsilon(K'_i)})$$
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$$\varepsilon(s) = s(N - s) \equiv \text{dispersion relation}$$
The partition function of the HS chain

Finkel & AGL PRB 2005

The partition function of the (antiferromagnetic) Haldane–Shastry $\text{su}(m)$ spin chain is given by

$$
Z(T) = \sum_{k \in \mathcal{P}_N} q^{r-1} \varepsilon(K_i) \prod_{i=1}^{r} \binom{m}{k_i} \prod_{i=1}^{N-r} \left(1 - q^\varepsilon(K'_i)\right)
$$

Example: $N = 7, m = 3, k = (1, 2, 1, 3)$
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Example: $N = 7$, $m = 3$, $\mathbf{k} = (1, 2, 1, 3)$

$$\implies r = 4, \quad (K_1, K_2, K_3) = (1, 3, 4), \quad (K'_1, K'_2, K'_3) = (2, 5, 6)$$
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$$\implies \binom{3}{1} \binom{3}{2} \binom{3}{1} \binom{3}{3} q^{1(7-1)+3(7-3)+4(7-4)} \times (1 - q^{2(7-2)}) (1 - q^{5(7-5)}) (1 - q^{6(7-6)})$$
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\Rightarrow r = 4, \quad (K_1, K_2, K_3) = (1, 3, 4), \quad (K'_1, K'_2, K'_3) = (2, 5, 6)
\]

\[
\rightarrow 27q^{30}(1 - q^{10})^2(1 - q^6)
\]
The partition function of the HS chain

Finkel & AGL PRB 2005

The partition function of the (antiferromagnetic) Haldane–Shastry $\text{su}(m)$ spin chain is given by

$$Z(T) = \sum_{\mathbf{k} \in \mathcal{P}_N} \sum_{r=1}^{\mathbf{k}} \prod_{i=1}^{r} \left( \frac{m_{K_i}}{k_i} \right) \cdot \prod_{i=1}^{N-r} \left( 1 - q^{E(K'_i)} \right)$$

- $K_i, K'_i \in \mathbb{N} \implies Z$ polynomial in $q \implies$ all the energies are nonnegative integers!
Outline of the talk

1. Quantum Calogero–Sutherland models
   - Scalar models
   - Spin CS models

2. Spin chains of Haldane–Shastry type
   - The HS chain
   - The freezing trick
   - The partition function

3. Quantum chaos
   - Basic concepts
   - Level density
   - Distribution of the spacings

4. Conclusions
Unfolding
**Unfolding**

**Idea:** characterize the degree of *integrability/chaos* of a *quantum* system through *statistical properties* of its spectrum

- Spectrum: $E_0 < E_1 < \cdots < E_n$, $d_i \equiv$ degeneracy of the level $E_i$
- Cumulative level density (normalized to 1):
  $$F(E) = \frac{1}{d} \sum_{i, E_i \leq E} d_i, \quad d \equiv \sum_{i=0}^{n} d_i \quad (= m^N)$$

- In order to compare spectra with different level densities, it is necessary to map the “raw” spectrum into a “normalized” spectrum with approximately constant level density $\rightarrow$ unfolding
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\[ \implies \langle s_i \rangle = 1 \]
The conjectures of Bohigas–Giannoni–Schmit and Berry–Tabor
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- In the Gaussian orthogonal ensemble (GOE) of random matrices, the probability density $p(s)$ of the spacing $s$ between consecutive levels of the unfolded spectrum approximately follows Wigner’s law

$$p(s) = \frac{\pi s}{2} e^{-\frac{\pi s^2}{4}}$$

\(\rightarrow\) level repulsion

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Figure 3.1. Nearest neighbour distance distribution for the so-called nuclear data ensemble. It contains altogether 1726 nuclear energy levels of 36 sequences of 32 different nuclei. The solid line corresponds to the Wigner distribution [Boh83] (with kind permission from Kluwer Academic Publishers).

Figure 2.11. Nearest neighbour distance histograms collected from a number of rectangularly shaped microwave resonators with side lengths $a = 16.5 \ldots 51.0$ cm, $b = 20$ cm in the two frequency ranges 5 to 10 GHz (a) and 15 to 18 GHz (b) [Haa91], and for the quarter stadium billiard shown in Fig. 2.10 (c) [Stö90] (Copyright 1990+91 by the American Physical Society).
The conjectures of Bohigas–Giannoni–Schmit and Berry–Tabor


In a quantum system (invariant under time reversal) whose classical limit is chaotic, the spacings distribution $p(s)$ should approximately follow Wigner’s law.
**Berry–Tabor conjecture (1977):**

The spacings distribution of a “generic” integrable quantum system should obey Poisson’s law

\[ p(s) = e^{-s} \]

\[ \rightarrow \text{uncorrelated (unfolded) levels} \]

- Examples: Heisenberg chain, $t-J$ model, Hubbard model [Poilblanc et al. EL 93], chiral Potts model [d’Auriac et al. JPA 02]
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From fig. 1-3 it is clear that there is a simple conclusion in accord with the speculations by Montambaux et al.; namely the distribution is Poisson for the integrable cases, GOE for the generic case. As the level distribution is observed to be independent of the choice of the subspace of total spin $S$, we take the largest subspace calculated. We remark that if we are

Fig. 1. – Level statistics for the Heisenberg chain with 20 spins. Ideal GOE and Poisson distributions are shown as full lines. The levels are in a single subspace of total spin $S = 2$ with 1280 levels and fixed total momentum 0. The second-nearest-neighbour coupling varies as: a) $J_2 = 0$, the integrable case; b) $J_2 = 0.25$; and c) $J_2 = 0.5$.

Fig. 2. – The $t$-$J$ model with one hole, $N = 16$ sites, minimal total spin $S = 1/2$ and coupling $J$ taking the values a) $J = 5$; b) $J = 2$, the integrable point; and c) $J = 1$. The total momentum is taken as $k = \pi/8$. 
Spin chains of HS type are excellent candidates for testing the validity of Berry–Tabor’s conjecture, since they are integrable models whose spectrum can be exactly computed for a very large number of degrees of freedom ($m^N$)
The level density of spin chains of HS type
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The first step for determining the spacings distribution of a spectrum is to find a continuous approximation \( \eta(E) \) to its cumulative level density \( F(E) \)
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Enciso, Finkel & AGL PRE 2010 ($A_{N-1}$ case)

As $N \to \infty$, the level density of all spin chains of HS type tends to a normal distribution having as parameters the mean $\mu$ and standard deviation $\sigma$ of the spectrum
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As $N \to \infty$, the level density of all spin chains of HS type tends to a normal distribution having as parameters the mean $\mu$ and standard deviation $\sigma$ of the spectrum

$$\Rightarrow \quad \eta(E) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{E} e^{-\frac{(E'-\mu)^2}{2\sigma^2}} \, dE' = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{E - \mu}{\sqrt{2\sigma}} \right) \right]$$
The level density of spin chains of HS type

Example: plot of the cumulative level density $F(E)$ vs. the cumulative gaussian distribution $\eta(E)$ for the antiferromagnetic HS chain.
The level density of spin chains of HS type

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\[ N = 20, \ m = 2 \]
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- **Example:** plot of the cumulative level density \( F(E) \) vs. the cumulative gaussian distribution \( \eta(E) \) for the antiferromagnetic HS chain

\[
F(E), \eta(E) \quad F(E), \eta(E)
\]

\[
N = 20, \ m = 2 \\
[ n = 2^{20} \approx 1.0 \times 10^6 ]
\]

\[
N = 15, \ m = 3 \\
[ n = 3^{15} \approx 1.4 \times 10^7 ]
\]
The spacings distribution of the PF and HS chains
The spacings distribution of the PF and HS chains

- In order to compute the spacings density $p(s)$, we must first apply to the raw spectrum the unfolding transformation

$$E_i \leftrightarrow \varepsilon_i = \eta(E_i) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{E_i - \mu}{\sqrt{2} \sigma} \right) \right]$$

- In the PF chain, the energy levels $E_i$ are equispaced:

$$E_i = E_0 + i \delta, \quad i = 0, \ldots, n; \quad \delta \equiv \text{spacing}$$

- In the HS chain, when $N \gg 1$ the raw spectrum is approximately equispaced, with predominant spacing $\delta = 1 + (N \mod 2)$
The spacings distribution of the PF and HS chains

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The spacings distribution of the PF and HS chains

Example: plot of $\delta_i \equiv E_{i+1} - E_i$ vs. the standardized energy $e_i \equiv (E_i - \mu) / \sigma$ for $N = 26$ and $m = 2$
The spacings distribution of the PF and HS chains

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It is possible to derive an excellent analytic approximation to the cumulative spacings density

$$P(s) \equiv \int_0^s p(s')ds'$$

of any spectrum possessing these two properties.
The spacings distribution of the PF and HS chains

- When \( N \gg 1 \), the spectra of the HS and PF chains have the following properties:
  - The level density is approximately Gaussian
  - The spectrum is (approximately) equispaced

**Key idea:** the above assumptions imply that

\[
S_i \sim \frac{n \delta}{\sqrt{2\pi \sigma}} e^{-\frac{(E_i - \mu)^2}{2\sigma^2}}
\]
The spacings distribution of the PF and HS chains

Final result [Barba, Finkel, AGL & Rodríguez PRB 08]

\[ P(s) \approx \frac{2}{\sqrt{\pi} s_{\text{max}}} \sqrt{\log(s_{\text{max}}/s)}, \]

with

\[ s_{\text{max}} = \frac{n \delta}{\sqrt{2\pi} \sigma}, \quad n \equiv \text{number of spacings} \]
The spacings distribution of the PF and HS chains

Final result [Barba, Finkel, AGL & Rodríguez PRB 08]

\[
P(s) \quad \sim \quad \frac{1}{N \gg 1} \quad 1 - \frac{2}{\sqrt{\pi} s_{\text{max}}} \quad \sqrt{\log(s_{\text{max}}/s)} ,
\]

with

\[
s_{\text{max}} = \frac{n \delta}{\sqrt{2\pi} \sigma} , \quad n \equiv \text{number of spacings}
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In particular, the spacings distribution \( p(s) \equiv P'(s) \) of the PF and HS chains is neither of Wigner’s nor of Poisson’s type!
The spacings distribution of the PF and HS chains

Example: plot of the cumulative spacings density $P(s)$ vs. its analytic approximation for the PF and HS chains with $m = 2$ and $N = 25$
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PF chain (156 different levels)
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PF chain (156 different levels)  

HS chain (562 different levels)
Outline of the talk

1 Quantum Calogero–Sutherland models
   - Scalar models
   - Spin CS models

2 Spin chains of Haldane–Shastry type
   - The HS chain
   - The freezing trick
   - The partition function

3 Quantum chaos
   - Basic concepts
   - Level density
   - Distribution of the spacings

4 Conclusions
Spin chains of Haldane–Shastry type are a class of integrable one-dimensional lattice models sharing several characteristic properties markedly different from those of other integrable chains:
Conclusions

- Each of them is derived from a corresponding **dynamical (CS)** model

- The **partition function** can be computed in closed form for arbitrary $N$

- The spectrum is **highly degenerate** and consists entirely of **integers**

- The level density is **normally distributed** for $N \gg 1$

- The spacings distribution follows a **square-root-of-a-logarithm** law, which is neither **Poisson's** (characteristic of other **integrable models**) nor **Wigner's** (typical of fully chaotic systems)

- More recently, it has been shown [Barba, Finkel, AGL & Rodríguez PRE 2009] that the **spectral noise** of the PF and HS chains has a power spectrum $S(k) \sim k^{-4}$, different from that of other **integrable models** ($k^{-2}$) or of chaotic systems ($k^{-1}$)
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