Optimal Portfolios and Random Matrices

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January 17, 2015
Overview

1. Background
2. Methods
3. Results
4. Conclusions
A basic method to minimize the risk when making an investing strategy is to use Markowitz Mean Variance Optimization:

$$\text{min} \frac{1}{2} w' \Sigma w \text{ subject to } 1' w = 1$$

It is well known that the portfolios obtained by this method are undiversified, unstable and have large short positions. This motivates to find improvements over this method.
Background

As we mentioned last Monday, the small eigenvalues come from the noise, yet they have the biggest impact in the optimal portfolio. We compare the eigenvalue distribution of the empirical correlation matrix and the random correlation matrix.

Figure: $\sigma^2$ Best fit = 0.527 ; $\lambda_{max} = 1.423$
We'll clean the noise from the empirical covariance matrix, and then obtain the optimal portfolio. We studied three methods:

- Bouchaud.
- Ledoit.
- EBIT/EV.
Bouchad: averaging the eigenvalues

The method:

- Replace all the noise-induced eigenvalues by their average.
- Solve MVO problem with new matrix.

The reason is that the optimal portfolio has the form

\[ w^* = \tilde{\mu} + \sum_{1 \leq i \leq n} (\lambda_i^{-1} - 1)(e_i' \tilde{\mu}) e_i \]

and therefore by averaging the noise-induced eigenvalues we make the solution more stable and we avoid over-weighting the \( e_i \).
Ledoit: Honey, I shrinked the covariance matrix

The method:

- Compute a highly structured matrix $F$ by the following the procedure:

$$
\Sigma \rightarrow C \rightarrow \begin{pmatrix}
1 & \bar{\text{cor}} & \bar{\text{cor}} \\
\bar{\text{cor}} & 1 & \bar{\text{cor}} \\
\bar{\text{cor}} & \bar{\text{cor}} & 1
\end{pmatrix} \rightarrow F
$$

- Produce modified covariance matrix

$$
\tilde{\Sigma} = \delta F + (1 - \delta)\Sigma
$$

where (non-trivial) statistical estimators are used to find the best $\delta$.

- Use $\tilde{\Sigma}$ to find the optimal portfolio.
The method:

- We add the constraint \( w_i \geq 0 \) for the top 25 companies (out of 50) when we rank them by their EBIT/EV.
- Solve the MVO problem with this additional constraint and find the optimal portfolio.

The reason for this constraint is to use other financial ratios to improve the reliability of the optimization. We want to avoid counter-intuitive portfolios. We can try this method for other financial ratios too.
We applied three versions of each method, and we compared with the MVO problem.

- Original version.
- Constraining to only long positions:
  \[ 0 \leq w \]
- Adding an upper bound:
  \[ 0 \leq w \leq 0.1 \]
Comparison of results

<table>
<thead>
<tr>
<th></th>
<th>MVO</th>
<th>Bouchaud</th>
<th>Ledoit</th>
<th>EBIT/EV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Return</td>
<td>0.0865</td>
<td>0.0812</td>
<td>0.0694</td>
<td>0.1017</td>
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<tr>
<td>Std</td>
<td>0.1515</td>
<td>0.1274</td>
<td>0.1309</td>
<td>0.1368</td>
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<tr>
<td>I.R.</td>
<td>0.5707</td>
<td>0.6371</td>
<td>0.5297</td>
<td>0.7436</td>
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<tr>
<td>VaR</td>
<td>-0.0600</td>
<td>-0.0469</td>
<td>-0.0556</td>
<td>-0.0554</td>
</tr>
<tr>
<td>CVaR</td>
<td>-0.0916</td>
<td>-0.0817</td>
<td>-0.0825</td>
<td>-0.0832</td>
</tr>
<tr>
<td>Divers</td>
<td>0.0765</td>
<td>0.0531</td>
<td>0.0552</td>
<td>0.0674</td>
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</tbody>
</table>

**Figure:** Original version
Comparison of results

<table>
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<tr>
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</thead>
<tbody>
<tr>
<td>Return</td>
<td>0.0967</td>
<td>0.0959</td>
<td>0.0862</td>
<td>0.0967</td>
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<tr>
<td>Std</td>
<td>0.1296</td>
<td>0.1271</td>
<td>0.1293</td>
<td>0.1296</td>
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<tr>
<td>I.R.</td>
<td>0.7462</td>
<td>0.7542</td>
<td>0.6665</td>
<td>0.7462</td>
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<tr>
<td>VaR</td>
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<td>-0.0492</td>
<td>-0.0548</td>
<td>-0.0493</td>
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<tr>
<td>CVaR</td>
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<td>-0.0825</td>
<td>-0.0836</td>
<td>-0.0833</td>
</tr>
<tr>
<td>Divers</td>
<td>0.0475</td>
<td>0.0475</td>
<td>0.0490</td>
<td>0.0475</td>
</tr>
</tbody>
</table>

Figure: $0 \leq w$
Comparison of results

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<tbody>
<tr>
<td>Return</td>
<td>0.1137</td>
<td>0.1096</td>
<td>0.1046</td>
<td>0.1137</td>
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<tr>
<td>Std</td>
<td>0.1342</td>
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<td>I.R.</td>
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<td>VaR</td>
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</tr>
<tr>
<td>CVaR</td>
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<tr>
<td>Divers</td>
<td>0.0301</td>
<td>0.0304</td>
<td>0.0305</td>
<td>0.0301</td>
</tr>
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</table>

Figure: $0 \leq w \leq 0.1$
In addition to the previous findings, we are interested in the eigenvalue distribution of the last two methods (Ledoit’s and EBIT/EV).

- Ledoit’s method produces a modified covariance matrix $\tilde{\Sigma}$.
- EBIT/EV method adds constraints to the MVO problem. Using Lagrange multipliers and KarushKuhnTucker conditions, it is possible to obtain an equivalent unconstrained problem with a modified matrix $\tilde{\Sigma}$.

We study the eigenvalue distribution of the modified matrices, and we obtain the weight of the meaningful section of eigenvalues.
Ledoit’s method

Figure: $\sigma^2 = 0.65, Q = 8.76, \text{Meaningful weight} = 0.44$
Ledoit’s method, global scale

The eigenvalues are concentrated, similar to Bouchaud’s method.

Figure: $\sigma^2 = 0.65, Q = 8.76, \text{Meaningful weight} = 0.44$
Figure: $\sigma^2 = 0.80, Q = 1.84, \text{Meaningful weight} = 0.39$
Figure: $\sigma^2 = 0.66, Q = 1.94, \text{Meaningful weight}=0.43$
Figure: $\sigma^2 = 0.66, Q = 1.90, \text{Meaningful weight} = 0.44$
For the original version of the methods, the evidence is in favor of Bouchaud.

Same for $w \geq 0$, with a very close MVO solution.

When putting an upper bound, the MVO solution provides better results.

Across all 12 versions, the best return is given by MVO 3.0, the best standard deviation by Bouchaud 2.0, the best I.R. by MVO 3.0, best VaR by Bouchaud 1.0, best CVaR by Bouchaud 1.0 and best diversification by MVO 3.0.

In terms of significant eigenvalues, the best method is EBIT/EV 3.0, but they are all close.
Future directions

- Try the equivalent of EBIT/EV method for other financial ratios.
Thanks!