The Lie operad and representation theory

Recent work-in-progress by Behrens and Rezk on TAQ (cf also [6]) has me rereading Arone and Dwyer's classic paper on partition complexes and Tits buildings:

The Goodwillie derivatives $\partial_* I$ of the identity functor on pointed spaces define an operad in spectra [2], with structure maps

$$\partial_n I \wedge \partial_{i_1} I \wedge \cdots \wedge \partial_{i_n} I \to \partial_{\mathbf{i}} I$$

(where $\mathbf{i} = \sum i_k$), equivariant with respect to the cabling homomorphism

$$\Sigma_n \times \Sigma_{i_1} \times \cdots \times \Sigma_{i_n} \to \Sigma_{\mathbf{i}} \;;$$

so its homotopy quotients define maps

$$(\partial_n I)_{h\Sigma_n} \wedge (\partial_{i_1} I)_{h\Sigma_{i_1}} \wedge \cdots \wedge (\partial_{i_n} I)_{h\Sigma_{i_n}} \to (\partial_i I)_{h\Sigma_{\mathbf{i}}}$$

of spectra. But by [1 Theorem 1.6],

$$H_*((\partial_n I)_{h\Sigma_n}, \mathbb{F}_p) \;=\; 0$$

unless $n = p^k, \; k \geq 1$: in which case the homology is isomorphic to a graded sum

$$\epsilon_k^{\mathrm{ST}} \, H_*(B\mathbb{F}_p^k, \mathbb{F}_p^{\pm}) \;:=\; V_*(k) \otimes \mathrm{St}_*(k)$$

of copies of the Steinberg representation of $\mathrm{Gl}_k(\mathbb{F}_p)$ (which acts on $\mathbb{F}_p^k$ as usual, and on $\mathbb{F}_p^{\pm}$ through the sign of the determinant; ϵ_k^{ST} is a Steinberg idempotent, and the asterisk on St means that it is a graded vector space (concentrated in degree $k-1$)).

Some time ago Kapranov suggested [4 §3.3] that certain natural transformations

$$\mathrm{St}(k) \otimes \mathrm{St}(l) \to \mathrm{St}(k+l)$$

might be analogous (in the hypothetical limit $\mathrm{Gl}_n(\mathbb{F}_p) \to \Sigma_n$ as $p \to 1$) to some vestige of an operad structure. [Thanks to him also for a reference to [3].]

If we regard $n = p^{k+l}$ as the sum of p^k copies of p^l, then the operad structure maps above define (unfamiliar to me) homomorphisms

$$\mathrm{Hom}(V^*(k) \otimes V^*(l)^{\otimes p^k}, V^*(k+l)) \to \mathrm{Hom}(\mathrm{St}_*(k) \otimes \mathrm{St}_*(l)^{\otimes p^k}, \mathrm{St}_*(k+l))$$

(where
$$V^*(k) \ := \ \mathrm{Hom}_{\mathrm{Gl}_k(\mathbb{F}_p)}(H_*(B\mathbb{F}_p^k, \mathbb{F}_p^{\pm}), \mathrm{St}_*(k))$$
is the graded dual[1] of $V_*(k)$). Could some natural transformation

$$\otimes^p V^*(l) \to V^*(l)$$

(perhaps induced by a homomorphism

$$H^*(B\mathbb{F}_p^{pl}) \to H^*(B\mathbb{F}_p^l)$$

defined eg by a base extension

$$\mathbb{F}_p^l \to \mathbb{F}_p^l \otimes_{\mathbb{F}_p} \mathbb{F}_{\mathbf{p}} \cong \mathbb{F}_p^{pl}$$

with $\mathbf{p} = p^p$) define something like an operad structure on the Steinberg modules?

Some references

1. G Arone, W Dwyer, Partition complexes, Tits buildings and symmetric products. Proc. London Math. Soc. (3) 82 (2001) 229 - 256

2. M Ching, Bar constructions for topological operads and the Goodwillie derivatives of the identity. Geom. Topol. 9 (2005), 833 - 933

3. T Foster, Higher configuration operads by way of quiver Grassmannians, `arXiv:1211.4525`

4. M Kapranov, Analogies between the Langlands correspondence and topological quantum field theory, in **Functional analysis on the eve of the 21st century** I 119 - 151, Progr. Math. 131, Birkhäuser (1995)

5. N Kuhn, S Mitchell, The multiplicity of the Steinberg representation of $\mathrm{Gl}_n(\mathbb{F}_q)$ in the symmetric algebra. Proc. AMS 96 (1986) 1 - 6

6. C Rezk, Modular isogeny complexes, Algebraic & Geometric Topology 12 (2012) 1373-1403, `arXiv:1102.5022`

[1] Kuhn and Mitchell considered closely related functors a while ago [4], and I suspect the technology for their study has advanced since then.