# Math 5286H

Midterm 3

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, April 9**.

1. Suppose an abelian group has the presentation matrix
$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 0 & 5 \\ 0 & 4 & 0 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}.$$
Express this group as a product of cyclic groups.

2. Given the matrix
$$A = \begin{bmatrix} (t-1) & (t^3-t) & t^2 \\ (t+1) & (t^2-1) & t^2 \end{bmatrix}$$
with coefficients in $\mathbb{R}[t]$, find invertible matrices $Q$ and $P$ in $\mathbb{R}[t]$ such that $QAP^{-1}$ is in a diagonal form
$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{bmatrix}$$
with $d_i$ monic polynomials such that $d_1$ divides $d_2$.

3. Suppose $R$ is a ring and $M$ is an $R$-module. If $m \in M$, the *annihilator* of $m$ is defined to be the set
$$Ann(m) = \{r \in R \mid rm = 0\}.$$
   (a) Show that $Ann(m)$ is an ideal of $R$, and $Ann(rm) \supseteq Ann(m)$ for all $r \in R$.

   (b) Suppose $0 \neq m \in M$ is an element with a *maximal annihilator*, meaning that for any other element $n \in M$ such that $Ann(n) \supseteq Ann(m)$, we have $Ann(n) = Ann(m)$. Show then that $Ann(m)$ is a *prime* ideal.

4. Show that any square complex matrix $A$ can be written in the form $D+N$, where:
   - $D$ is diagonalizable,
   - $N$ is nilpotent, i.e. $N^k = 0$ for some $k$, and
   - $D$ commutes with $N$, i.e. $DN = ND$.

5. In a *noncommutative* ring, there is a distinction between *left* ideals (subgroups $I \subset R$ such that, for all $r \in R$ and $x \in I$, $rx \in I$) and right ideals (which ask instead that $xr \in I$).

   The two-by-two matrix ring $M_2(\mathbb{R})$ is noncommutative. Give an example of a subgroup $I \subset M_2(\mathbb{R})$ which is a left ideal but not a right ideal.