Math 5286H

Midterm 2 Solutions

1. (a) For any ring R with an ideal $I \subsetneq R$, prove that there is a bijective correspondence between maximal ideals of R/I and maximal ideals of R that contain I.

   Solution. The correspondence theorem gives a bijection between the ideals of R/I and the ideals of R containing I, so it suffices to show that this correspondence preserves maximal ideals. This correspondence sends an ideal J of R to the ideal J/I of R/I, and the third isomorphism theorem says $R/J \cong \frac{R/I}{J/I}$. The ideal J is maximal if and only if R/J is a field, and J/I is maximal if and only if $\frac{R/I}{J/I}$ is a field. Therefore, maximality of the two is equivalent.

   (b) Find all maximal ideals of the ring $\mathbb{C}[x,y]/(x^5 - 8y^2, y-2)$.

   Solution. By the previous part, these are the same as maximal ideals of $\mathbb{C}[x,y]$ containing $x^5 - 8y^2$ and $y-2$. By the Nullstellensatz, all such ideals are of the form $(x-a, y-b)$ for complex numbers a, b such that $a^5 - 8b^2 = 0$ and $b-2 = 0$. This forces $b = 2$ and $a^5 = 32$, which has exactly 5 roots: $a = 2e^{2\pi ik/5}$ for $k \in \{0,1,2,3,4\}$. Therefore, there are 5 maximal ideals, all of the form $(x - 2e^{2\pi ik/5}, y-2)$.

2. Suppose that R is an integral domain, $\mathbb{F}$ is a field, and $\phi : R \to \mathbb{F}$ is a ring homomorphism. Let K be the field of fractions of R. Show that there exists a homomorphism $\phi' : K \to \mathbb{F}$ such that $\phi'(r/1) = \phi(r)$ for all $r \in R$ if and only if $ker(\phi) = 0$.

   Solution. If ϕ is an injection, then by chapter 10, proposition 6.7 in Artin, such a homomorphism ϕ' exists. Conversely, suppose ϕ' exists. Then for any nonzero element $r \in R$ we have:

   $$1 = \phi'(1/1) = \phi'(r/r) = \phi'(r/1)\phi'(1/r) = \phi(r)\phi'(1/r)$$

   and hence we cannot have $\phi(r) = 0$ because $1 \neq 0$ in $\mathbb{F}$. Therefore, ϕ has no kernel and hence is injective.

3. Let $S \subset \mathbb{C}^2$ be the set of elements $\{(0,n) | n \in \mathbb{Z}\}$.

   (a) Determine the ideal $I(S) \subset \mathbb{C}[x,y]$ of functions vanishing on S.

   Solution. The polynomial x vanishes on S, and so $(x) \subset I(S)$. Conversely, suppose $f(x,y) = \sum a_{ij}x^i y^j$ is a polynomial vanishing on S. Then $f(0,y) = \sum a_{0j} y^j$ is a polynomial in y with zeros at all integers. The only polynomial with infinitely many zeros is the zero polynomial, and so $f(0,y) \equiv 0$. This implies that all the nonzero terms in $f(x,y)$ are divisible by x, and so $f(x,y) \in (x)$. Therefore, $I(S) = (x)$.

(b) Determine the variety $V(I(S)) \subset \mathbb{C}^2$.

Solution. This variety is the set of points that are zeros of all elements of $I(S) = (x)$. The set of zeros of the polynomial x is precisely the x-axis $\{(0, y) | y \in \mathbb{C}\} \subset \mathbb{C}$, so $V((x))$ is contained inside the x-axis. Conversely, any element of (x) is of the form $f(x, y) = xg(x, y)$ for some g, and so it vanishes on the x-axis. Therefore, $V((x))$ is equal to the x-axis.

4. Factor the following elements into primes in the following unique factorization domains.

(a) $x^8 + x$ in $\mathbb{Z}/2[x]$.

Solution. This factors as $x(x + 1)(x^3 + x^2 + 1)(x^3 + x + 1)$. All these polynomials are cubic and none have any roots, so they are irreducible.

(b) $12 + 24i$ in $\mathbb{Z}[i]$.

Solution. We have $12+24i = (-1)\cdot 3\cdot (1+i)^4 \cdot (1+2i)$. The elements 3, $(1 + i)$, and $(1 + 2i)$ are irreducible.

(c) $4x^3 + 18x + 6$ in $\mathbb{Z}[x]$.

Solution. This factors as $2 \cdot (2x^3 + 9x + 3)$. 2 is irreducible in $\mathbb{Z}$ and hence in $\mathbb{Z}[x]$, whereas $(2x^3 + 9x + 3)$ is primitive and irreducible either by Eisenstein's criterion using the prime 3, or by the rational zeros theorem by checking the possible roots $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$.

(d) $x^3 - y^3$ in $\mathbb{Q}[x, y]$.

Solution. This factors as $(x - y)(x^2 + xy + y^2)$. These are both primitive in polynomials with coefficients in the unique factorization domain $\mathbb{Z}[y]$, and so it suffices to check that they are irreducible in the fraction field $\mathbb{Q}(y)$. The first one is linear and hence irreducible, whereas the latter is irreducible because it has no roots in $\mathbb{Q}(y)$.

5. A prime ideal P of R is called *minimal* if there are no prime ideals Q with $0 \subsetneq Q \subsetneq P$.

(a) If R has factorization into irreducibles and $a \in R$ is a prime element, show that the ideal (a) is a minimal prime ideal.

Solution. Suppose $P \subset (a)$ is a prime ideal. Either $P = 0$ or there is a nonzero element $x \in P$. The element x can be factored as $x = x_1 \cdots x_r$ where the x_i are irreducible, by assumption. Since P is prime and $x_1 \cdots x_r \in P$, one of the factors x_i must be in $P \subset (a)$. However, we then have that a divides x_i, so $x_i = ab$ for some $b \in R$. As a is not a unit (units cannot generate prime ideals) and x_i is irreducible, this forces b to be a unit. Therefore, $a = x_i b^{-1} \in P$, so $(a) = P$.

(b) If R is a UFD, show conversely that every minimal prime ideal is of the form (a) for some prime element $a \in R$.

Solution. Suppose P is a minimal prime ideal. There exists some nonzero element $y \in P$. Since R is a UFD, y factors into primes: $y = y_1 \cdots y_s$. Since P is prime and $y_1 \cdots y_s \in P$, one of the factors y_j must be in P. But then $(y_j) \subset P$ is a nonzero prime ideal contained in P, and so by minimality $(y_j) = P$.